

Oscillatory Integrals and Curved Kakeya Sets

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To Chris.

Abstract

This thesis investigates analogues for curves of the *Keakeya* conjecture for straight lines in \mathbb{R}^n . These arise from Hörmander's conjecture about oscillatory integrals in the same way as the straight line case comes from the Restriction and Bochner-Riesz problems. The problem is to determine from the phase what the minimal dimension for the corresponding *curved Keakeya set* is. This is defined to be a set which includes a translate of each member of a specified collection of curved arcs. For the straight line case, the minimum is conjectured to be n , but some curves are known to admit *Keakeya sets* with dimension as low as $\frac{n+1}{2}$. We focus almost entirely on parabolic curves, since the corresponding question about quadratic phases is the simplest for which such examples are known.

First of all we prove that such sets of curves can indeed have zero measure. Then we show that the lower bound of $\frac{n+1}{2}$ holds for all of the families of quadratic curves under consideration. We then use both geometric and arithmetic techniques (which were developed for the straight line case by Bourgain, Wolff, Katz and Tao) and apply them to the curved case. In this way we are able to obtain lower bounds such as $\frac{4n+3}{7}$ for *Keakeya sets* consisting of particular types of curves, and results for the more difficult maximal function problem corresponding to the bound $\frac{n+2}{2}$. The curves for which such bounds can be proved are specified in terms of algebraic criteria for matrices occurring in their coefficients.

Interestingly, the best results we obtain are for Nikodym sets of the very same curves for which no bound greater than $\frac{n+1}{2}$ for the *Keakeya* version is possible. The thesis ends with a discussion of the insights this brings into the relationship between *Keakeya* and Nikodym phenomena.

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*You seem to have an enormous memory for useless facts.
And most of what I tell you is useless, and is not a fact.
Tony Carbery*

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List of Notation

$ \cdot $	Lebesgue measure	9
\bar{V}	Closure of V	9
\mathbb{B}^{n-1}	Unit ball of \mathbb{R}^{n-1}	10
ω	Usually an element of \mathbb{R}^{n-1}	10
x'	(x_1, \dots, x_{n-1})	10
\mathcal{H}^s	Hausdorff measure	11
\dim	Dimension (Minkowski or Hausdorff)	11
nb_{δ}	Delta-neighbourhood	11
\mathcal{K}_{δ}	Takeya maximal function	12
	Curved Takeya maximal function	25
\mathcal{N}_{δ}	Nikodym maximal function	12
	Curved Nikodym maximal function	25
$\mathbb{1}_E$	Characteristic (or indicator) function of a set E	13
C_c^{∞}	Set of infinitely differentiable functions with compact support ...	15
σ	Surface measure on the unit sphere	15
$*$	Convolution	15
J_n	The Bessel function of order n	16
\hat{f}	Fourier transform of f	15
$\text{Ball}(x, r)$	The ball in \mathbb{R}^n centre x radius r	15
\lesssim	Less than, but for a constant and perhaps a log	16
ε_j	Random signs ± 1	17
x	Usually an element of \mathbb{R}^n	21
y	Usually an element of \mathbb{R}^{n-1}	21
$a(x, y)$	C_c^{∞} cutoff function	21
$\varphi(x, y)$	Phase function	21
T_N	Oscillatory integral operator	21
$\Gamma_y(\omega)$	Parabolic curve centre ω and direction y	24
$T_y^{\delta}(\omega)$	Delta-tube around $\Gamma_y(\omega)$	24
$\mathcal{L}\mathcal{K}_{\delta} \mathcal{L}\mathcal{N}_{\delta}$	Linearised maximal functions	26
Q_j	Delta-cube belonging to a partition of \mathbb{B}^{n-1}	26

$\#$	Cardinality of a finite set	30
t	Usually an element of $[-1, 1]$	31
A, B	$(n - 1) \times (n - 1)$ matrices occurring as coefficients in φ	31
tr	The trace of a matrix	32
${}^m\mathbb{Z}^n$	$n \times m$ matrices, considered as m -tuples of vectors in \mathbb{R}^n	40
$[p]$	The natural numbers $1, 2, \dots, p$	45

Chapter 1

The straight line problems

You can have lots of good ideas while walking in a straight line.
Jonathan Bennett

The main focus of this thesis is an investigation of curved analogues of a well-known conjecture, namely the *Keakeya* problem. This deals with families of straight lines and how they overlap. It is still far from being solved, but much progress has been made in the last fifteen years, and so we shall begin by explaining what the problem is and how it has been attacked.

Although the *Keakeya* conjecture is of great importance in several branches of mathematics, the curved analogue considered here arises via its link with harmonic analysis. So we shall explain how the straight line case becomes important in harmonic analysis by looking at three famous problems. Then in the following chapter we explain the relevance of the curved case and what is already known about it. Finally we shall state and prove a number of preliminary results relating to curves which we use continually in the rest of the thesis.

The main elements of this chapter and the next are summarised in diagrams on pages 20 and 36.

1.1 *Keakeya* and Nikodym sets

The *Keakeya* problem is named after the Japanese mathematician who in 1917 asked the following purely geometric question: What is the set in the plane of smallest area that allows a needle of unit length to be reversed? That is, a unit line segment can, by rotating and sliding, be returned to its original position but the other way round without having left the set. Clearly a disc of unit diameter would do, but an equilateral triangle of unit height is smaller. Smaller still is the three-cusped hypocycloid, in which the needle can perform a typical three-point turn. For a time many believed this to be optimal, but in 1920 Besicovitch proved

the surprising result that the area can be made arbitrarily small. (1928 English version [1].)

In doing this he discovered an even more extraordinary fact: There exists a set in the plane that includes a unit line segment in every direction yet has area zero. This is true despite the fact that such a set includes uncountably many lines. It is these sets rather than the needle-reversal ones that are nowadays named after Kakeya and on which much attention has recently been focused, although in the literature one sometimes sees them called Besicovitch sets in honour of their discoverer.

The original proof (and some clearer versions due to Perron and Schoenberg [28, 32]) relied on simple geometry involving triangles, together with an iterative procedure. For example, the account given in Falconer's book [14] begins by cutting up a triangle and repeatedly sliding adjacent pieces so as to preserve the directions of the line segments while reducing the area by causing overlap. A different proof was given much later by Kahane [18], which used arithmetic in the form of quaternary expansions of numbers. He showed that if two Cantor sets are joined by line segments in a certain way then the result is a Kakeya set of measure zero.

Meanwhile, Nikodym was studying the opposite type of problem—sets that include line segments passing through many points—in response to a question asked by Banach in connexion with differentiation theorems. Nikodym proved [27] that there exists a null set in the unit square such that every point of the complement is “linearly accessible through the set”, which means it lies on a line that is otherwise included in the set. However, we prefer a slightly different but related definition, and so for us a *Nikodym set* in the plane will be roughly one that includes a unit line segment, whose direction is at most $\pi/4$ from the vertical, through every point on the interval $[-1, 1]$. These sets can have zero area. Again, the original proof was an iterative one using only geometry, but an arithmetic approach also works; in fact, Kahane's set described above has both the Kakeya and Nikodym properties at the same time.

The use of both arithmetic and geometric methods pervades the subject and will be seen throughout this thesis, even when dealing with curves and in higher dimensions. It has even been remarked that if these two approaches could be unified, it might lead to a resolution of the conjecture.

Because of our modification of the definition and the complicated nature of the original proof given by Nikodym, we give here a simple geometric construction in the spirit of the well-known Perron tree proof of the existence of measure zero Kakeya sets.

Theorem 1. *Nikodym sets as defined above can have zero area.*

Proof: The main idea is to construct sets of arbitrarily small area that include all the required segments, and then use a compactness argument to get the area equal to zero.

Start with a triangle of unit height sitting on the unit interval. Clearly this has the Nikodym property. If the base is divided into many intervals and hence the triangle into many thin triangles, then these may be sheared by moving their apex horizontally in such a way as to make the area small.

To make this precise we first study what happens when we shear two adjacent triangles. Start with two adjacent but disjoint triangles, each of base length b and height h . Then shear the triangles by moving their apices horizontally in any manner so that they cross over by a distance $2\alpha b$ with $0 < \alpha < 1$, to obtain a shape like that in Figure 1.1.

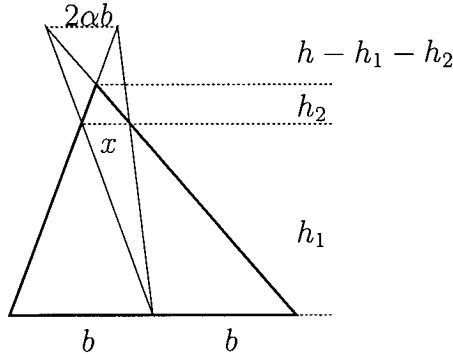


Figure 1.1: *Two sheared triangles*

By similar triangles it follows that

$$\frac{x}{h - h_1} = \frac{b}{h} \qquad \frac{2b}{h_1 + h_2} = \frac{2\alpha b}{h - h_1 - h_2} \qquad \frac{2\alpha b}{h} = \frac{x}{h_1}.$$

Solving these shows that the overlap of the triangles has area $\frac{\alpha}{(1+\alpha)(1+2\alpha)}bh$ and that the area of the new triangle (outlined in bold) is $\frac{1}{1+\alpha}bh$.

Now, let us start with a triangle of base B and height 1, and divide it along the base into 2^{n+1} thin triangles, where both n and α are chosen later. The total area $A_0 = B/2$. Perform the shearing above on adjacent pairs, so that the total area is reduced by $\frac{\alpha}{(1+\alpha)(1+2\alpha)}A_0$, and the area in the bold triangles is $\frac{1}{1+\alpha}A_0$. Now repeat this process, but shear pairs of adjacent bold triangles as shown in Figure 1.2.

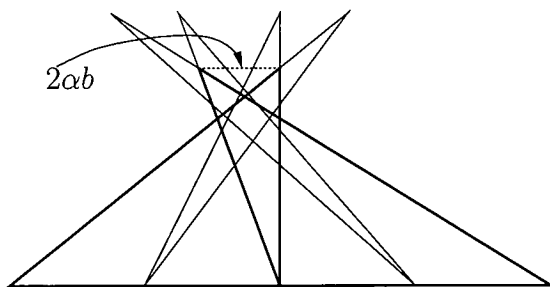


Figure 1.2: *The second stage of shearing*

We obtain successive areas as follows:

$$\begin{aligned}
A_0 &= \frac{1}{2}B \\
A_1 &= A_0 - \frac{\alpha}{(1+\alpha)(1+2\alpha)}A_0 \\
A_2 &\leq A_0 - \frac{\alpha}{(1+\alpha)(1+2\alpha)}A_0 - \frac{\alpha}{(1+\alpha)(1+2\alpha)}\frac{1}{1+\alpha}A_0 \\
&\vdots \\
A_n &\leq A_0 - \frac{\alpha}{(1+\alpha)(1+2\alpha)}A_0 - \frac{\alpha}{(1+\alpha)(1+2\alpha)}\frac{1}{1+\alpha}A_0 - \dots \\
&\quad \dots - \frac{\alpha}{(1+\alpha)(1+2\alpha)}\left(\frac{1}{1+\alpha}\right)^{n-1}A_0 \\
&= A_0\left(\frac{2\alpha}{1+2\alpha} + \frac{1}{1+2\alpha}\left(\frac{1}{1+\alpha}\right)^{n-1}\right)
\end{aligned}$$

which may be made arbitrarily small by simply choosing α small, and then n huge.

So we have constructed a set of arbitrarily small area which includes unit line segments through every point of the unit interval. Note that during the construction, the apex of each triangle moved at most B at each step, of which there were n . So if we wish, we can carry out the construction within some open set that includes our starting triangle, by simply dividing up the triangle to make the new starting base B extremely small. (This is allowable since the n required depended on α but not on B .)

We are now ready for the final compactness argument. This follows Falconer's argument for the Kakeya version in [14]. Let S_1 be an isosceles triangle of unit base and height, and let V_1 be an open set including S_1 such that $|\bar{V}_1| \leq 2|S_1|$. By what we have just shown, we can divide and shear the triangle to form a figure $S_2 \subseteq V_1$ with area at most 2^{-2} . Then, since we are only talking about finite unions of triangles, we can fit in another open set V_2 so that $S_2 \subseteq V_2 \subseteq V_1$ and $|\bar{V}_2| \leq 2|S_2|$. Now divide and shear each of the triangles making up S_2 to obtain a figure $S_3 \subseteq V_2$ of area at most 2^{-3} .

Continuing in this way we obtain an infinite sequence of figures such that $S_i \subseteq V_i \subseteq V_{i-1}$ and $|\bar{V}_i| \leq 2|S_i| \leq 2^{1-i}$. Let $E = \bigcap_{i=1}^{\infty} \bar{V}_i$. We claim that E is a Nikodym set of measure zero. The measure zero part is obvious, so it remains to show that E includes a unit line segment through each point of the base interval. Clearly this is true of each \bar{V}_i , so for a fixed point on the base interval, let l_i be a line segment through it that is included in \bar{V}_i . By compactness, some subsequence of the top endpoints of the l_i converges, and so the subsequence of segments converges to some segment l through the given point. Since the \bar{V}_i are closed and nested, $l \subseteq E$ as required. \square

Higher dimensionalakeya and Nikodym sets are defined similarly: Aakeya set in \mathbb{R}^n is a set that includes a unit line segment in every direction, that is, parallel to every element of the sphere S^{n-1} , while a Nikodym set includes a line segment, with angle at most $\pi/4$ from the vertical, through each point of \mathbb{B}^{n-1} , the unit ball of \mathbb{R}^{n-1} . For technical reasons discussed later, we define the Nikodym set to be only those parts that lie outside a fixed neighbourhood of the hyperplane $x_n = 0$. Such sets can obviously have measure zero: simply take the Cartesian product of an example in \mathbb{R}^2 with $[-1, 1]^{n-2}$, to produce a cylinder. This includes all the required segments, and has zero measure by Fubini's theorem.

We should note here thatakeya and Nikodym sets may be transformed one into the other by the map $(x', x_n) \mapsto (x'/x_n, 1/x_n)$ used in [7]. (Here our removal of a neighbourhood of $x_n = 0$ from the Nikodym set corresponds to theakeya lines having finite length.) More explicitly, the straight line $x' = \omega + x_n y$ which is in direction $(y, 1)$ and passes through $(\omega, 0)$ gets mapped to the line $x' = x_n \omega + y$, in which the roles of direction and centre are exchanged. For this reason the more recent literature has been entirely devoted to the former type of set, since the two are equivalent. We have chosen to distinguish them because of important differences appearing when considering curved analogues.

1.2 Dimension and maximal conjectures

So far we have talked about the size of these sets in terms of measure, and explained that families of lines can be made to lie in surprisingly small sets. Theakeya and Nikodym conjectures, on the other hand, state that such sets cannot be too small. Precisely, it is conjectured that bothakeya and Nikodym sets in \mathbb{R}^n must have full dimension.

There are many notions of dimension for sets in \mathbb{R}^n , all of which agree with the obvious integer values when the set in question is, say, a smooth piece of surface. Two of the notions have relevance here, namely Hausdorff dimension (perhaps

the most widely used), and Minkowski dimension (which is simpler to work with but lacks some of the desirable properties of the Hausdorff dimension). We now give these two concepts precise formulation.

Definition (Hausdorff Dimension). Let $E \subseteq \mathbb{R}^n$ and let $s > 0$. Define the s -dimensional Hausdorff measure of E by

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \inf \left\{ \sum_{i=0}^{\infty} (\text{diam } U_i)^s : E \subseteq \bigcup_{i=1}^{\infty} U_i, 0 < \text{diam } U_i \leq \delta \right\}.$$

Then the Hausdorff dimension of the set E is the value $\dim E := \sup\{s > 0 : \mathcal{H}^s(E) > 0\}$ with the convention that $\dim E = 0$ if the set on the right is empty.

It can be shown that in fact \mathcal{H}^s is a Borel measure which agrees with Lebesgue measure (at least up to a constant multiple) for integer s . The dimension is thus the critical value where the s -dimensional measure of the set changes from being infinite to being zero, and agrees with the familiar idea of dimension for smooth sets. Further details are contained in many textbooks such as [14].

Definition (Minkowski dimension). Let $E \subseteq \mathbb{R}^n$ and let $\text{nb}_\delta(E) := \{x \in \mathbb{R}^n : \text{dist}(x, E) \leq \delta\}$ be its δ -neighbourhood. Then the (upper) Minkowski dimension of E is the value

$$\dim E := \sup \left\{ s > 0 : \limsup_{\delta \downarrow 0} \frac{|\text{nb}_\delta(E)|}{\delta^{n-s}} > 0 \right\}$$

with the same convention as before. (Recall that $|\cdot|$ denotes Lebesgue measure.)

Since $\frac{|\text{nb}_\delta(E)|}{\delta^{n-s}}$ is comparable to δ^s times the number of balls of diameter δ needed to cover E , it is easily seen that the Minkowski dimension must be greater than or equal to the Hausdorff dimension. Strict inequality is possible (consider the rationals) but the Minkowski dimension does agree with the usual notion for smooth sets. Mattila's book [25] may be consulted for further information.

We can now state the Kakeya problem more precisely: It says that a Kakeya set (or equivalently a Nikodym set) in \mathbb{R}^n must have both Minkowski and Hausdorff dimension n . We have also seen that the latter assertion is stronger, since the Hausdorff dimension may be less than the Minkowski. Note also that had we not excluded a neighbourhood of $x_n = 0$, then Nikodym sets would trivially have dimension at least $n - 1$, with no corresponding obvious bound for Kakeya sets.

A still stronger form of the conjecture may be given in terms of maximal functions. Such functions are of great importance in harmonic analysis and it is mainly this version of the conjecture that we focus on.

Definition (Takeya maximal function). Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ and $\delta > 0$. The Takeya maximal function of f is $\mathcal{K}_\delta f : S^{n-1} \rightarrow \mathbb{R}$ defined by

$$\mathcal{K}_\delta f(\theta) := \sup_{T \parallel \theta} \frac{1}{|T|} \int_T |f(x)| dx,$$

where T denotes a tube of length 1 and width δ with centre in the plane $x_n = 0$.

So this operator returns the largest possible average of f over tubes of fixed direction. Often we shall take only directions within $\pi/4$ of the vertical, which is enough since we could always consider rotated copies of this.

The Nikodym maximal function is similar, except it is the position of the tube that is fixed and the direction that is varied.

Definition (Nikodym maximal function). Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ and $\delta > 0$. The Nikodym maximal function of f is $\mathcal{N}_\delta f : \mathbb{B}^{n-1} \rightarrow \mathbb{R}$ defined by

$$\mathcal{N}_\delta f(\omega) := \sup_{T \ni (\omega, 0)} \frac{1}{|T|} \int_T |f(x)| dx,$$

where T is as before and has direction at most $\pi/4$ from the vertical.

To confuse matters further, many authors have referred to this operator by the name Takeya, which is unfortunate since this may also have clouded the interesting symmetry between the Restriction and Bochner-Riesz problems as discussed later. However our usage agrees with Bourgain who introduced the curved analogue, and is in accordance with the naming of the associated sets.

The precise shape of the tubes, whether rounded or cuboidal, is irrelevant here. The properties of the operator depend on the eccentricity. The conjecture is that both operators map $L^n \rightarrow L^n$ with norm at most $C\delta^{-\varepsilon}$ —that is, the norm grows more slowly than any power of $1/\delta$.

Trivially, the $L^1 \rightarrow L^\infty$ norm is of order $\delta^{-(n-1)}$, while the $L^\infty \rightarrow L^\infty$ norm is 1. Also, since S^{n-1} and \mathbb{B}^{n-1} are compact, every bound into L^p automatically holds for L^q for all smaller q . These facts together with the Riesz interpolation theorem give conjectures for a range of p, q as follows:

Conjecture 1 (Takeya and Nikodym maximal conjectures). *The Takeya and Nikodym maximal functions satisfy*

$$\begin{aligned} \|\mathcal{K}_\delta\|_{L^p(\mathbb{R}^n) \rightarrow L^q(S^{n-1})} &\leq C_\varepsilon \delta^{1-n/p-\varepsilon} \\ \|\mathcal{N}_\delta\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{B}^{n-1})} &\leq C_\varepsilon \delta^{1-n/p-\varepsilon} \end{aligned}$$

for $1 \leq p \leq n$ and $1 \leq q \leq (n-1)p'$.

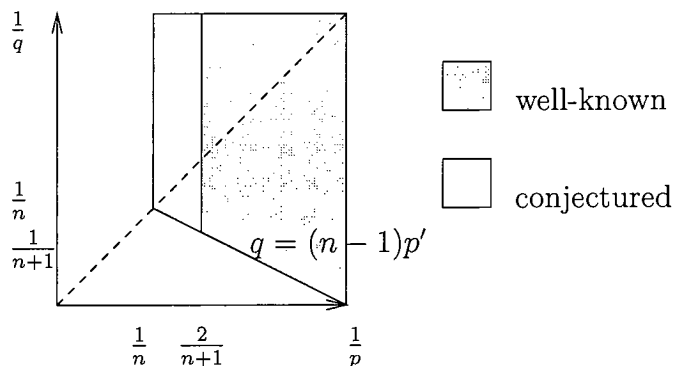


Figure 1.3: *Region where the $L^p \rightarrow L^q$ norms of the Keakeya and Nikodym maximal operators are at most $\delta^{1-n/p-\epsilon}$*

These ranges of exponents are summarised in the usual $(1/p, 1/q)$ diagram (Figure 1.3). The example where f is the characteristic function of a single tube shows that there can be no decrease with δ in the $L^p \rightarrow L^q$ norm, and so the region is best possible. Again, these two conjectures are equivalent by the transformation described on page 10.

Maximal conjectures are stronger than their “set dimension” counterparts. This is very easy to see in the case of the Minkowski dimension:

Proposition 2 (Maximal function implies Minkowski dimension). *Suppose that $\|\mathcal{K}_\delta\|_{p \rightarrow q} \leq C\delta^{-\alpha}$ on \mathbb{R}^n . Then the Minkowski dimension of every Keakeya set in \mathbb{R}^n is at least $n - p\alpha$.*

Proof: Let $K \subseteq \mathbb{R}^n$ be a Keakeya set, and let $f = \mathbb{1}_{\text{nb}\delta K}$. Then $\|f\|_p = |\text{nb}\delta K|^{1/p}$. By definition, $\text{nb}\delta K$ includes a $1 \times \delta$ tube in every direction, so that $\mathcal{K}_\delta f \equiv 1$ and hence $\|\mathcal{K}_\delta f\|_q$ is a constant. So the hypothesis tells us that $|\text{nb}\delta K| \geq C\delta^{p\alpha}$, and so

$$\limsup_{\delta \downarrow 0} \frac{|\text{nb}\delta K|}{\delta^{n-s}}$$

is positive if $s < n - p\alpha$ as claimed. \square

The proof for Nikodym is of course analogous. The implication for the Hausdorff dimension is also true, but we omit the proof here since we shall give a detailed proof for the curved case in Theorem 11.

1.3 Known partial results

Before looking at the wider implications of these problems in analysis and considering the curved case, let us review the progress to date.

The conjecture is known to be true in two dimensions: For the Keakeya set version, this was shown in 1971 by Davies [13], and for the maximal function

(in Nikodym form) by Córdoba [12]. In higher dimensions the first results were due to Christ, Duoandikoetxea and Rubio de Francia in the 80s [11] with a lower bound of $\frac{n+1}{2}$ for the dimension of the sets, and the corresponding result for the maximal function ($p = \frac{n+1}{2}$).

In 1991 Bourgain [2] showed that in \mathbb{R}^3 every Kakeya set has dimension at least $7/3$ and gave similar small improvements over $\frac{n+1}{2}$ in higher dimensions, as well as proving the corresponding maximal function results; these were obtained by geometric methods. Further geometry was used by Wolff [40] to obtain $\frac{n+2}{2}$, again for all versions of the conjecture.

Greater progress in higher dimensions occurred with the introduction of arithmetic techniques. The first paper, of Bourgain [5], gave lower bounds of the form $\alpha(n-1) + 1$ for some $\alpha > 1/2$ with some maximal function results. The idea was quickly improved by Katz and Tao [21] to give $\frac{6n+5}{11}$ for the Hausdorff dimension and $\frac{4n+3}{7}$ for the Minkowski. They have extended these ideas several times, and the best results in higher dimensions to date are lower bounds of $0.5969n + 0.403$ (approximately) for the Minkowski dimension and $(2 - \sqrt{2})(n-4) + 3$ for the Hausdorff dimension of straight-line Kakeya sets, and a sharp $p \rightarrow q$ result with $p = \frac{4n+3}{7}$ for the Kakeya maximal function [23]. These are currently the best known for large enough n .

Jointly with Łaba, meanwhile, they tackled lower dimensions (where Wolff's bound had not been superseded), and in 2000 with great effort were able to improve it slightly to $\frac{n+2}{2} + 10^{-n}$ [20, 24]. These papers used the ideas of Wolff together with more sophisticated geometric methods, introducing notions such as “stickiness”, “planiness” and “graininess”.

The main purpose of this thesis is to apply some of these techniques to the curved analogue, and hence gain an insight not only into what is true for curves, but also what special properties of straight lines must be used in order to have hope of solving the main conjecture.

1.4 The link with Harmonic analysis

The problems described above are not merely interesting for their own sake; they have links with many other important parts of mathematics. These include number theory (Montgomery's conjecture, and via it, the Riemann hypothesis), measure theory (Furstenberg's conjecture, Falconer's distance set conjecture), combinatorics (the size of sumsets, which we look at in Chapter 6; and via them multilinear operators) and PDE, via harmonic analysis. References for all these are contained in [22, 41, 42]. Because it is in the latter context that the curved

case arises, we outline the link with harmonic analysis now.

1.4.1 The ball multiplier

The first link occurred in C. Fefferman's classic paper [16] on the multiplier problem for the ball. This problem arises naturally in considering inversion of the Fourier transform in dimensions two and above. One asks whether for an L^p function f , the “partial sums”

$$\int_{|\xi| < R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

converge to f in L^p , which is equivalent to asking whether the ball multiplier $\widehat{Tf} = \mathbb{1}_{\text{Ball}(0,1)} \hat{f}$ is L^p bounded. If the characteristic function was that of a rectangle rather than a ball, then we would have boundedness for all p , since such an operator may be built out of Hilbert transforms as described in [33]. The latter book mentions the ball as an unsolved problem, but a year after its publication, Fefferman surprised everyone by showing that T is only bounded on L^p for $p = 2$.

The counterexample he gave involved the construction of a Nikodym-type set; essentially the proof shows that if the ball multiplier was bounded, then Nikodym sets could never have zero measure. This is the usual direction of the implications: There is some operator in harmonic analysis of interest, and its boundedness would imply that Kakeya/Nikodym sets are large to some quantified extent.

1.4.2 Bochner-Riesz

In some sense, the pathology of the ball multiplier is that the characteristic function of the ball is anything but smooth. Replacing the characteristic function by a nicer cutoff is roughly equivalent (via scaling and partitions of unity) to considering a smooth bump function supported on an annulus. This is the Bochner-Riesz problem. It may be formulated as follows: Let χ be a positive C_c^∞ function supported on the annulus $1 - \delta < |\xi| < 1$ and define the multiplier operator $\widehat{T_\delta f}(\xi) := \chi(\xi) \hat{f}(\xi)$. What, in terms of n and p , is the best power of δ that majorises $\|T_\delta\|_{p \rightarrow p}$?

The answer is believed to be $\frac{n+1}{2} - \frac{n}{p}$ for $1 \leq p < \frac{2n}{n+1}$ with of course 0 in the trivial case $p = 2$, and $\frac{n}{p} - \frac{n-1}{2}$ for $\frac{2n}{n-1} < p \leq \infty$ following by interpolation and duality. To see that no better power is possible, set f to be a Schwartz function such that $\hat{f} \equiv 1$ on the unit ball. Then $\widehat{T_\delta f} = \chi$ so it follows that $\|T_\delta\|_{p \rightarrow p} \gtrsim \|\chi\|_p$. Now χ is just $\delta \sigma * \phi_\delta$ where ϕ is a normalised bump function, σ is surface measure on the sphere and $\phi_\delta := \delta^{-n}(\cdot/\delta)$ is its dilation, preserving the total mass. It is a

standard result (see e.g. [36]) that

$$\hat{\sigma}(x) = \frac{2\pi}{|x|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|x|) \sim |x|^{-\frac{n-1}{2}}$$

for large x . Meanwhile, $\widehat{\phi_\delta}(x) = \hat{\phi}(\delta x)$ which is essentially $\mathbb{1}_{\{x:|x|<1/\delta\}}$. Hence we calculate that $\|\hat{\chi}\|_p = C\delta^{\frac{n+1}{2}-\frac{n}{p}}$ for $p \leq \frac{2n}{n-1}$. This shows that the powers stated above are best possible.

These bounds are known to be true in two dimensions [9], and in higher dimensions with the additional assumption that p lies outwith the interval $[\frac{2(n+1)}{n+3}, \frac{2(n+1)}{n-1}]$ [15].

As with the ball multiplier, good behaviour of the Bochner-Riesz multiplier would show that Nikodym sets must be large. In particular the best result at the endpoint $p = \frac{2n}{n-1}$ would imply that Nikodym sets have dimension n . This implication is already known, but since the proofs in the literature have invariably reformulated in terms of Kakeya sets first, we include a sketch of a direct proof here.

In view of our non-standard definition of Nikodym sets we use an $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ version of the Bochner-Riesz operator. This is given by

$$\widehat{S_\delta f}(\xi) = \chi(\xi)\hat{f}(\xi').$$

Note that if the usual Bochner-Riesz operator has norm A from L^p to itself, then this version has norm $\delta^{1/p'}A$. Also, as in the rest of the thesis, we shall prove a *restricted weak type estimate*, that is, estimate the norm of the operator from the Lorentz space $L^{p,1}$ to $L^{p,\infty}$. This is enough to give the claimed results by applying the Marcinkiewicz interpolation theorem. In general, to prove that some operator T has restricted weak type p, q norm A , one has to show that

$$|\{x : |T\mathbb{1}_E(x)| \geq \lambda\}| \leq \left(\frac{A|E|^{\frac{1}{p}}}{\lambda} \right)^q$$

for all sets E of finite measure and all $\lambda > 0$.

Theorem 3. *Suppose that $\|S_\delta f\|_p \lesssim \delta^{-(\frac{n-1}{2}-\frac{n}{p})+\frac{1}{p'}}\|f\|_p$. Then with $r = (p/2)'$, the Nikodym maximal function is of restricted weak type (r, r) with norm at most $\delta^{-2(n/r-1)}$.*

Proof: Write the norm as $K[\delta]$. We begin by showing that

$$\left\| \sum_{j=1}^M \mathbb{1}_{T_j} \right\|_{p/2} \leq K[\delta](\delta^{n-1}M)^{2/p}$$

for every collection $\{T_j\}_{j=1}^M$ of $1 \times \delta$ tubes whose intersections with the plane $x_n = 0$ are δ -separated. This is a covering lemma as in [6]; Similar statements are discussed in more detail later (Lemma 10). By rescaling and redefining δ appropriately, we see that this is equivalent to

$$\left\| \sum_{j=1}^M \mathbb{1}_{T_j} \right\|_{p/2} \leq K[\sqrt{\delta}](\delta^{-\frac{n+1}{2}} M)^{2/p}$$

for tubes of size $\frac{1}{\delta} \times \frac{1}{\sqrt{\delta}}$ whose points at height 0 are $\delta^{-1/2}$ -separated. (Note that we always have “ K ” of the eccentricity, times what would be obtained if the tubes had been disjoint.) This rescaling is done so that the “dual rectangle” T_j^* of T_j is of size $\delta \times \delta^{1/2}$ and so, after a translation to \tilde{T}_j^* , fits perfectly into the annulus. (Recall that the Fourier transform of the characteristic function of a rectangle in \mathbb{R}^n is essentially an oscillation times the measure of the rectangle times the characteristic function of the “dual rectangle” that has side lengths equal to the reciprocals of the side lengths of the original.)

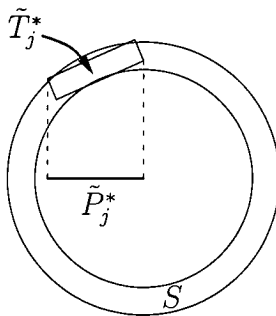


Figure 1.4: \tilde{T}_j^* and its projection

Now since T_j is at an angle of at most $\frac{\pi}{4}$ from the vertical, its dual is at most $\frac{\pi}{4}$ from the horizontal, so that the projection \tilde{P}_j^* of \tilde{T}_j^* onto $\xi_n = 0$ is essentially a cube of side length $\sqrt{\delta}$. More precisely, after a suitable translation to P_j^* say, this projection is the dual of some set P_j , where $|P_j| \sim \delta^{\frac{n-1}{2}}$ and the set of P_j has bounded overlap.

Suppose that we set $\hat{f}(\xi') = \sum \varepsilon_j \mathbb{1}_{\tilde{P}_j^*}$, where the $\pm \varepsilon_j$ are random signs. Then $f(x')$ is essentially $\delta^{\frac{n-1}{2}} \sum \varepsilon_j e^{ia_j} \mathbb{1}_{P_j}$ for some suitable translations a_j . (To make this rigorous, we could smooth the characteristic functions to C_c^∞ functions, whose Fourier transforms then decay rapidly.) Since the sets P_j have bounded overlap it follows that $\|f\|_p \lesssim \delta^{\frac{n-1}{2}} (\delta^{-\frac{n-1}{2}} M)^{1/p}$.

On the other hand, $\widehat{S_\delta f}(\xi) = \sum \varepsilon_j \mathbb{1}_{\tilde{T}_j^*}$ so that $S_\delta f(x) = \delta^{\frac{n+1}{2}} \sum \varepsilon_j e^{ib_j} \mathbb{1}_{T_j}$ for some translations b_j . Now use Khinchin's inequality. This says that if ε_j are random ± 1 s, a_j are complex numbers and $t > 0$ then

$$(\mathbb{E} |\sum_j \varepsilon_j a_j|^t)^{1/t} \sim (\sum_j |a_j|^2)^{1/2}.$$

Applying this with $t = 1$ gives

$$\|S_\delta f\|_p \gtrsim \delta^{\frac{n+1}{2}} \left\| \sum_{j=1}^M \mathbb{1}_{T_j} \right\|_{p/2}^{1/2}$$

which rearranges to give the covering lemma above.

It is now easy to obtain the restricted weak type estimate. Let $E \subseteq \mathbb{R}^n$ be a set of finite measure and let $\Omega = \{\omega \in \mathbb{R}^{n-1} : \mathcal{N}_\delta \mathbb{1}_E(\omega) \geq \lambda\}$. Pick a maximal δ -separated subset $\{\omega_j\}_{j=1}^M \subseteq \Omega$, so that $|\Omega| \sim \delta^{n-1}M$, and for each j find a δ -tube T_j passing through the point $(\omega_j, 0)$ such that $|T_j \cap E| \geq \lambda|T_j|$. Then

$$\begin{aligned} \lambda|\Omega| &\lesssim \sum_j |T_j \cap E| \\ &\lesssim |E|^{1-2/p} \left\| \sum \mathbb{1}_{T_j} \right\|_{p/2} \\ &\lesssim |E|^{1-2/p} |\Omega|^{2/p} \delta^{-2(n-1)+4n/p} \end{aligned}$$

as required. \square

Note in particular that the optimal exponent $p = \frac{2n}{n-1}$ implies the Kakeya conjecture, but because of the factor of 2 in the power of δ the results become very poor away from the optimal exponent. For example, when $p = \frac{2(n+1)}{n-1}$, which is known to be true, we have $r = \frac{n+1}{2}$ but deduce only the uninteresting information that Kakeya sets have dimension at least 1.

1.4.3 Restriction

We have now seen how the behaviour of Bochner-Riesz multipliers seems to be governed by the size of Nikodym sets. The harmonic analysis problem that corresponds to Kakeya is the *Restriction problem*. This is so named because in effect it asks when we can define the restriction to, say, a sphere, of the Fourier transform of an L^p function. One might have expected such restriction to have no meaning, since the sphere has measure zero and the Fourier transform of an L^p function is defined only almost everywhere. However, the curvature of the sphere makes this possible: For example, one can calculate the Fourier transform $\hat{\sigma}$ of the surface measure on the sphere in terms of Bessel functions as on page 16, and see that it decays like $O(|\xi|^{-\frac{n-1}{2}})$. This gives

$$\begin{aligned} \|\hat{f}\|_{L^2(d\sigma)}^2 &= \int \bar{\hat{f}} \hat{f} d\sigma \\ &= \int \bar{f}(f * \hat{\sigma}) && \text{by Parseval's inequality} \\ &\leq \|f\|_{L^p(\mathbb{R}^n)} \|f * \hat{\sigma}\|_{L^{p'}(\mathbb{R}^n)} \\ &\leq \|f\|_p^2 \|\hat{\sigma}\|_s && \text{by Young's inequality, } \frac{1}{p'} = \frac{1}{p} + \frac{1}{s} - 1 \end{aligned}$$

which is finite for $p < \frac{4n}{3n+1}$. So in this case the restriction does have some meaning, due to the decay of the Fourier transform of a measure supported on a curved surface. This simple example is far from optimal however, and the Restriction conjecture says that an estimate like

$$\|\hat{f}\|_{L^q(d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

holds if and only if $\frac{1}{q} \geq \frac{n+1}{n-1} \frac{1}{p'}$ and $\frac{1}{p} > \frac{n+1}{2n}$. The first condition is necessary, as can be seen by letting \hat{f} be a smooth bump function on a small “coin” of width δ and thickness δ^2 which fits nicely into a cap on the sphere. For the second condition, apply the adjoint operator (the *extension* operator)

$$\begin{aligned} g &\mapsto \mathcal{E}g, & L^{q'}(S^{n-1}) &\rightarrow L^{p'}(\mathbb{R}^n) \\ \mathcal{E}g(\xi) &:= \int_{S^{n-1}} g(\theta) e^{2\pi i \xi \cdot \theta} d\theta \end{aligned}$$

to the function $g \equiv 1$ and use the known decay of the Fourier transform of surface measure on the sphere.

The conjecture is true in dimension $n = 2$ [15], and for $q = 2$ in higher dimensions [39].

As promised, there is a link between these estimates and the size of Kakeya sets:

Theorem 4. *Suppose that $\|\hat{f}\|_{L^q(d\sigma)} \leq \|f\|_{L^p(\mathbb{R}^n)}$. Then, setting $r = \frac{p}{2-p}$, $s = \frac{q}{2-q}$, we have $\|\mathcal{K}_\delta\|_{L^s(S^{n-1})} \lesssim \delta^{-2(n/r-1)} \|f\|_{L^r(\mathbb{R}^n)}$.*

We do not prove this here since it is covered by the more general result that we shall prove in detail for curves (Theorem 8). Let us simply note that, again, the best possible result at the endpoint would imply the truth of the Kakeya conjecture, but away from the endpoint the implied bound for the dimension of Kakeya sets becomes very weak very quickly.

This chapter now ends with a diagram summarising the relationships between the straight line conjectures.

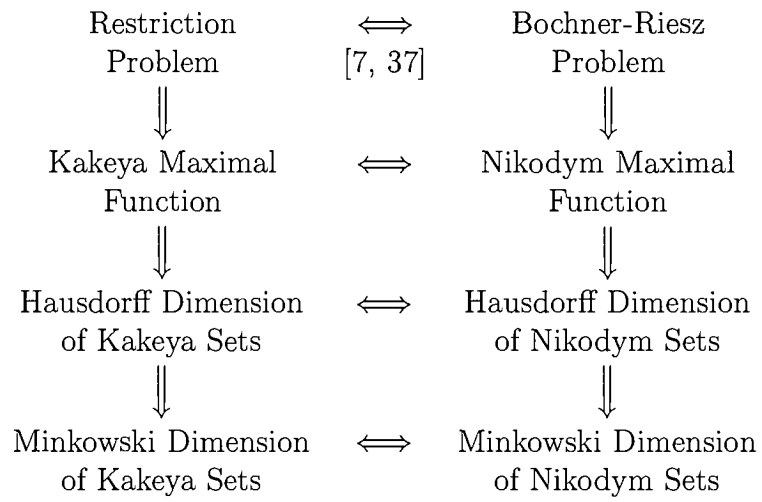


Figure 1.5: *The hierarchy of conjectures for the straight line problems*

Chapter 2

Background to the curved case

I don't like straight lines, no curvature. . .
Jim Wright

In this chapter we introduce the problems we seek to address in the rest of the thesis. These are curved analogues of the Keakeya and Nikodym problems, in both the set dimension and maximal versions.

But first we show how they arose out of harmonic analysis problems generalising the Restriction and Bochner-Riesz conjectures already described. Then we discuss what is already known about the curved case, and about the analytic problems that led to it. Finally we formulate precisely the curved analogues of Keakeya and Nikodym maximal functions, prove some very simple properties, and also gather together a few lemmas that will be used repeatedly in several parts of the thesis.

2.1 Hörmander's conjecture, and Bourgain's answer

The generalisation to curves of the Keakeya and Nikodym problems arises, unsurprisingly, from a generalisation of the Restriction and Bochner-Riesz problems. These two problems have been known to be linked for some time and are believed to be essentially equivalent [7, 37], just as the Keakeya and Nikodym conjectures are equivalent for straight lines. The generalisation we have in mind is a problem posed by Hörmander [17] concerning oscillatory integrals of the form

$$T_N f(x) := \int_{\mathbb{R}^{n-1}} e^{iN\varphi(x,y)} a(x,y) f(y) \, dy.$$

Here $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$, a is some smooth cut-off, and the phase function φ is assumed to be smooth on the support of a and to have the following properties:

$$\text{The matrix } \frac{\partial^2 \varphi}{\partial x \partial y}(x,y) \text{ has full rank } n-1. \tag{2.1}$$

For all $\theta \in S^{n-1}$ the map $y \mapsto \theta \cdot \frac{\partial \varphi}{\partial x}(x, y)$
has only non-degenerate critical points. (2.2)

The operator T_N is called an *oscillatory integral of the second kind* [35] and its norm from L^p to L^q is studied. Clearly as N increases, the oscillatory factor in the integrand will cause more cancellation, so the operator norm will decrease with N .

Both the Bochner-Riesz and Restriction operators can be formulated as special cases of the operator T_N . For Restriction this is not hard to see: In fact its adjoint, the extension operator mentioned previously, is already of this form if we simply parametrise the sphere by a bounded region of \mathbb{R}^{n-1} . This gives $\varphi(x, y) = x' \cdot y + x_n \sqrt{1 - |y|^2}$. Restriction to more general surfaces can be similarly formulated, but one thing all restriction problems have in common is that the phase function φ is linear in x . In particular for a paraboloid we have the very simple phase $\varphi(x, y) = x' \cdot y + x_n y^\top y$.

Reformulation of the Bochner-Riesz problem is more complicated and relates to an asymptotic representation of the Fourier transform of the measure supported on the surface. If the surface is given as a graph $(t, \gamma(t))$ with $t \in \mathbb{R}^{n-1}$ then the phase is roughly given by $\psi(x' - y, x_n)$, where

$$\psi(\xi) = \xi' \cdot (\nabla \gamma)^{-1}(-\xi'/\xi_n) + \xi_n \gamma((\nabla \gamma)^{-1}(-\xi'/\xi_n)).$$

For the paraboloid, this means that $\varphi(x, y) = |x' - y|^2/x_n$, which is essentially $\frac{1}{x_n}(x' \cdot y + y^\top y)$ since the term involving only x simply multiplies the integral by a factor with modulus 1.

Of course it must be verified that the criteria (2.1), (2.2) are satisfied by these phases. Note also that in the reformulation of Bochner-Riesz, the cutoff function a is assumed to vanish near $x_n = 0$, again showing the correctness of our removal of this hyperplane from the Nikodym set.

The similarity of the Restriction and Bochner-Riesz conjectures tempted Hörmander to ask whether a similar result might hold for the whole family of T_N . That is:

Conjecture 2. For $\frac{1}{q} < \frac{n-1}{2n}$, $\frac{1}{q} \leq \frac{n-1}{n+1} \frac{1}{p'}$ and φ satisfying (2.1), (2.2) is

$$\|T_N f\|_q \lesssim N^{-n/q} \|f\|_p ? \quad (2.3)$$

Hörmander himself proved this for $n = 2$ [17], and in higher dimensions it has been proved for $q \geq \frac{2(n+1)}{n-1}$ by Stein [34]. The known and conjectured regions are shown in Figure 2.1.

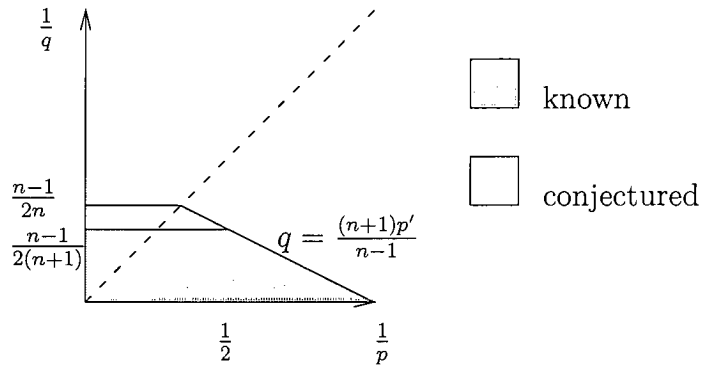


Figure 2.1: *Exponents for Hörmander's conjecture*

It was a great surprise in 1991 when Bourgain [3] disproved Hörmander's conjecture. Roughly, he showed that in dimension three for most phases the best exponent q is strictly greater than $\frac{2n}{n-1} = 3$, and that there exist phases where the known value $\frac{2(n+1)}{n-1} = 4$ is the best. More precisely:

Theorem 5 (Worst Case,[3]). *In dimension three there is a phase function, namely*

$$\varphi(x, y) = x_1 y_1 + x_2 y_2 + 2x_3 y_1 y_2 + x_3^2 y_1^2$$

for which (2.3) fails for all $q < 4$, even with $p = \infty$.

Proof: First check that this φ does indeed meet the criteria.

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \begin{pmatrix} 1 & 0 & 2y_2 + 4x_3 y_1 \\ 0 & 1 & 2y_1 \end{pmatrix}$$

which certainly has rank 2, while

$$\begin{aligned} \theta \cdot \frac{\partial \varphi}{\partial x} &= \theta_1 y_1 + \theta_2 y_2 + \theta_3 (2y_1 y_2 + 2x_3 y_1^2) \\ \frac{\partial}{\partial y} \left(\theta \cdot \frac{\partial \varphi}{\partial x} \right) &= \begin{pmatrix} \theta_1 + 2\theta_3 (y_2 + 2x_3 y_1) \\ \theta_2 + 2\theta_3 y_1 \end{pmatrix} \\ \frac{\partial^2}{\partial y^2} \left(\theta \cdot \frac{\partial \varphi}{\partial x} \right) &= \begin{pmatrix} 4x_3 & 2\theta_3 \\ 2\theta_3 & 0 \end{pmatrix} \end{aligned}$$

and so the first and second derivatives are both zero only if $\theta = 0 \notin S^{n-1}$.

Now apply the operator T_N to the function given by $f(y) = e^{iN y_2^2} \chi(y)$ where χ is some smooth cutoff, so that

$$T_N f(x) = \int_{\mathbb{R}^2} e^{iN(x_1 y_1 + x_2 y_2 + (y_2 + x_3 y_1)^2)} a(x, y) dy.$$

Set $(z_1, z_2) := (y_1, y_2 + x_3 y_1)$ and consider the surface $S := \{x : x_1 = x_2 x_3\}$. For $x \in S$ we have

$$\begin{aligned} T_N f(x) &= \int_{\mathbb{R}^2} e^{iN(x_2 z_2 + z_2^2)} \tilde{a}(x, z) dz_1 dz_2 \\ &= \int_{\mathbb{R}} e^{iN(x_2 z_2 + z_2^2)} \tilde{\tilde{a}}(x, z_2) dz_2 \end{aligned}$$

Now the phase has one non-degenerate critical point, at $z_2 = -\frac{1}{2}x_2$, so by the Stationary Phase Lemma (or direct calculation, since this is essentially $\int_{-1}^1 e^{iNx^2} dx$) we find that $|T_N f(x)| \sim 1/\sqrt{N}$ for $x \in S$. Also, $|\nabla_x T_N f(x)| = CN|T_N f(x)| \lesssim \sqrt{N}$ for $x \in \mathbb{R}^3$. Therefore we still have the estimate $|T_N f(x)| \geq 1/\sqrt{N}$ for $x \in \text{nb}_{1/N}(S)$. Hence

$$\|T_N f\|_q \geq \frac{C}{\sqrt{N}} |\text{nb}_{1/N}(S)|^{1/q} \sim N^{-\frac{1}{2}-\frac{1}{q}}$$

$$\|f\|_p \sim C.$$

So $\|T_N\|_{p \rightarrow q} \lesssim N^{-\frac{3}{q}}$ cannot be true unless $\frac{3}{q} \leq \frac{1}{2} + \frac{1}{q}$, which means that $q \geq 4$. \square

Theorem 6 (Generic Failure,[3]). *If φ has the property that*

$$\left. \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 \varphi}{\partial x_3^2} \right) \right|_{x=0, y=0} \text{ is not a multiple of } \left. \frac{\partial^2}{\partial y^2} \left(\frac{\partial \varphi}{\partial x_3} \right) \right|_{x=0, y=0} \quad (2.4)$$

then the inequality (2.3) cannot hold even for $p = \infty$ unless $q \geq 118/39 > 3$.

Theorem 5 extends easily to all odd dimensions, but the situation may be different in even dimensions as we shall see in Chapter 7. Theorem 6 will be proved later in the current chapter for a special class of curves. Having done that, we would like to know what information about φ is needed in order to determine the range of boundedness of the corresponding T_N . Bourgain's paper also contains a result showing that most phases do allow some non-trivial bound, so a complete understanding of the problem would mean being able to predict from the phase what the critical exponent will be. This thesis scratches the surface of this problem.

2.2 Oscillatory integrals imply Kakeya for curves

Bourgain's work showed that just as Restriction and Bochner-Riesz problems give rise to questions about sets of straight lines, so more general oscillatory integrals problems lead to questions about sets of curves. Given a phase function φ , define curves and curved tubes as follows:

Notation. Let $y \in \mathbb{B}^{n-1}$ be a direction, $\omega \in \mathbb{B}^{n-1}$ a centre and $\delta > 0$ a thickness. Define

$$\Gamma_y(\omega) := \{x \in \mathbb{R}^n : \nabla_y \varphi(x, y) = \omega, |x_n| < 1\}$$

$$T_y^\delta(\omega) := \{x \in \mathbb{R}^n : |\nabla_y \varphi(x, y) - \omega| < \delta, |x_n| < 1\}$$

to be the curve centre ω in direction y and the corresponding δ -tube.

The condition $|x_n| < 1$ is not significant: All that matters is that the range should be fixed and finite. In practice it should be chosen to match the support of the cutoff function a .

If we assume that our phase has the form

$$\varphi(x, y) = x' \cdot y + \tilde{\varphi}(x_n, y) \quad (2.5)$$

that is, its higher order terms depend only on x_n and not on x' , then the curve can be easily parametrised as

$$\Gamma_y(\omega) = \left\{ \begin{pmatrix} \omega - \nabla_y \tilde{\varphi}(x_n, y) \\ x_n \end{pmatrix} : x_n \in [-1, 1] \right\}.$$

In the special case where φ is linear in x , we may write $\varphi(x, y) = x' \cdot y + x_n \tilde{\varphi}(y)$ and find that the curve becomes a straight line through $\begin{pmatrix} \omega \\ 0 \end{pmatrix}$ whose direction depends only on y .

The tubes have the following crucial property, which will often allow us to consider only finite collections of tubes:

Lemma 7 (Engulfing property). *Suppose that $|y - \bar{y}| < \delta$ and $|\omega - \bar{\omega}| < \delta$. Then $T_y^{C\delta}(\omega) \supset T_{\bar{y}}^\delta(\bar{\omega})$ for some constant C depending only on φ .*

Proof: Since φ is smooth, we have $|\varphi(x, y) - \varphi(x, \bar{y})| < C\delta$ for some $C > 0$. By definition $x \in T_{\bar{y}}^\delta(\bar{\omega})$ means that $|\varphi(x, \bar{y}) - \bar{\omega}| < \delta$. Hence

$$\begin{aligned} |\varphi(x, y) - \omega| &< |\varphi(x, y) - \varphi(x, \bar{y})| + |\varphi(x, \bar{y}) - \bar{\omega}| + |\bar{\omega} - \omega| \\ &< C\delta + \delta + \delta = (C + 2)\delta. \end{aligned} \quad \square$$

By analogy with the straight line case, we define sets and maximal functions as follows:

Definition (Curved Kakeya and Nikodym sets). A set $E \subset \mathbb{R}^n$ is a curved Kakeya set (associated to φ) if for all $y \in \mathbb{B}^{n-1}$ there exists an $\omega \in \mathbb{B}^{n-1}$ such that $\Gamma_y(\omega) \subset E$. It is a curved Nikodym set associated to φ if for all $\omega \in \mathbb{B}^{n-1}$ there exists a $y \in \mathbb{B}^{n-1}$ such that $\Gamma'_y(\omega) \subset E$, where we define Γ' to be the part of Γ that remains after deletion of the fixed neighbourhood of $x_n = 0$.

Definition (Curved maximal functions). The curved Kakeya maximal function (associated to φ and of eccentricity $1/\delta$) is the operator that takes a function f on \mathbb{R}^n to the function $\mathcal{K}_\delta f$ on \mathbb{B}^{n-1} given by

$$\mathcal{K}_\delta f(y) := \sup_{\omega \in \mathbb{B}^{n-1}} \frac{1}{|T_y(\omega)|} \int_{T_y(\omega)} |f(x)| \, dx.$$

Similarly the curved Nikodym maximal function is given by

$$\mathcal{N}_\delta f(\omega) := \sup_{y \in \mathbb{B}^{n-1}} \frac{1}{|T_y(\omega)|} \int_{T_y(\omega)} |f(x)| \, dx.$$

The following theorem relates these definitions to the operators T_N , and the proof will show why they are natural.

Theorem 8. *Suppose that $\|T_N f\|_q \lesssim N^{-n/q} \|f\|_p$. Then the curved Keakeya maximal function is of restricted weak type (r, s) with norm at most $\delta^{-2(n/r-1)}$, where $r = (q/2)'$ and $s = (p/2)'$,*

To prove this, and also to prove the estimates for the maximal functions in later chapters, it is helpful to linearise the maximal function so that instead of an L^p bound we can prove a “covering lemma” analogous to those in [6].

Definition (Linearised operators). Decompose \mathbb{B}^{n-1} into disjoint δ -cubes Q_j for $j \in \mathbb{B}^{n-1} \cap \delta\mathbb{Z}^{n-1}$.

- (i) To each index j associate a tube $T_j = T_{y_j}^\delta(\omega_j)$ where $y_j \in Q_j$ and $\omega_j \in \mathbb{B}^{n-1}$. Define a linearisation of \mathcal{K}_δ by

$$\mathcal{L} \mathcal{K}_\delta f(y) := \sum_j \mathbb{1}_{Q_j}(y) \frac{1}{|T_j|} \int_{T_j} f(x) dx.$$

- (ii) To each index j associate a tube $T_j = T_{y_j}^\delta(\omega_j)$ where instead $\omega_j \in Q_j$ and $y_j \in \mathbb{B}^{n-1}$. Define a linearisation of \mathcal{N}_δ by

$$\mathcal{L} \mathcal{N}_\delta f(\omega) := \sum_j \mathbb{1}_{Q_j}(\omega) \frac{1}{|T_j|} \int_{T_j} f(x) dx.$$

It is enough for us to find bounds for these operators that are independent of the choices of the tubes T_j .

Lemma 9 (Equivalence of linearised versions). $\|\mathcal{K}_\delta\|_{L^p \rightarrow L^q} \leq A(\delta)$ for all choices of the tubes T_j if and only if $\|\mathcal{L} \mathcal{K}_\delta\|_{L^p \rightarrow L^q} \leq CA(C\delta)$, and a similar statement is true with \mathcal{N}_δ and $\mathcal{L} \mathcal{N}_\delta$.

Proof: First observe that whenever $|y - \bar{y}| < \delta$ we have $\mathcal{K}_\delta f(\bar{y}) \leq C^{n-1} \mathcal{K}_{C\delta} f(y)$. This is because $T_y^{C\delta}(\omega) \supset T_{\bar{y}}^\delta(\omega)$ by Lemma 7, and the larger tube has volume C^{n-1} times that of the smaller.

Now given $y \in \mathbb{B}^{n-1}$, let k be such that $y \in Q_k$. Regardless of the choice of the tubes T_j we have

$$\mathcal{L} \mathcal{K}_\delta f(y) = \mathcal{L} \mathcal{K}_\delta f(y_k) \leq \mathcal{K}_\delta f(y_k) \leq C^{n-1} \mathcal{K}_{C\delta} f(y)$$

which proves the “only if” part. For the “if” part, assume that the bound is true for all possible linearisations. Given f , choose a particular linearisation with the

property that for each j , the ω_j has been chosen to maximise the average of f over tubes in direction y_j . Then for this linearisation

$$\mathcal{K}_\delta f(y) \leq C^{n-1} \mathcal{K}_{C\delta} f(y_k) = C^{n-1} \mathcal{L} \mathcal{K}_{C\delta} f(y_k) = C^{n-1} \mathcal{L} \mathcal{K}_{C\delta} f(y)$$

as required.

The proof for the Nikodym version is similar, noting that Lemma 7 works with ω as well as with y . \square

Lemma 10 (Covering lemma). *Let $\{T_j\}_{j=1}^M$ be $1 \times \delta$ -tubes with centres ω_j and directions y_j (where both of these are in \mathbb{B}^{n-1}). Then the estimate*

$$\left\| \sum_{j=1}^M \mathbb{1}_{T_j} \right\|_{r'} \leq K(\delta^{n-1} M)^{1/s'}$$

holds for all choices of $y_j \in Q_j$ with arbitrary ω_j if and only if the (curved) Keakeya maximal function is of weak type (r, s) with constant K . Similarly, exactly the same estimate holds but instead $\omega_j \in Q_j$ with y_j arbitrary if and only if the Nikodym maximal function has this weak type estimate.

Proof: Observe that the adjoint of the linearised curved Keakeya maximal operator is given by

$$\mathcal{L} \mathcal{K}_\delta^* g(x) := \sum_j \mathbb{1}_{T_j}(x) \frac{1}{|T_j|} \int_{Q_j} g(y) dy$$

taking functions on \mathbb{B}^{n-1} to functions on \mathbb{R}^n . Now $\mathcal{L} \mathcal{K}_\delta$ is weak type (r, s) if and only if its adjoint is restricted type (s', r') with the same constant. Also the image of \mathcal{K}_δ consists of functions that are constant on cubes Q_j . So take $g = \mathbb{1}_{\bigcup_{j \in J} Q_j}$ where J is some index set of cardinality M . Then $\|g\|_{s'} = |\bigcup_{j \in J} Q_j|^{1/s'} = (\delta^{n-1} M)^{1/s'}$. Meanwhile

$$\|\mathcal{L} \mathcal{K}_\delta^* g\|_{r'} = \left\| \sum_{j \in J} \mathbb{1}_{T_j} \frac{|Q_j|}{|T_j|} \right\|_{r'} = \left\| \sum_{j \in J} \mathbb{1}_{T_j} \right\|_{r'}$$

so the result follows. The proof for the Nikodym case is the same. \square

Now we use the linearised versions to give a proof of Theorem 8, which will show the reason for the definition of the curves. This proof is similar to that given by Wolff in [42, pp. 153–154] for the Restriction problem, but incorporating ideas found in Bourgain’s “generic failure” proof for curves [3, pp. 326–327].

Proof of Theorem 8: Suppose that we are given tubes T_j with directions $y_j \in Q_j$ and arbitrary centres ω_j as above. Set

$$f(y) = \sum_{j=1}^M \varepsilon_j e^{-iN\omega_j \cdot (y-y_j)} \mathbb{1}_{Q_j}(y)$$

where the ε_j are random signs. By the disjointness of the cubes, $\|f\|_p = (\delta^{n-1}M)^{1/p}$. On the other hand

$$\begin{aligned} T_N f(x) &= \sum \varepsilon_j \int_{Q_j} e^{iN(\varphi(x,y) - \omega_j \cdot (y - y_j))} a(x, y) \, dy \\ &:= \sum \varepsilon_j I_j(x). \end{aligned}$$

Now $\varphi(x, y) = \varphi(x, y_j) + (y - y_j) \cdot \nabla_y \varphi(x, y_j) + O(|y - y_j|^2)$, so that if $x \in T_j$ and $y \in Q_j$ it follows that

$$\varphi(x, y) - \omega_j \cdot (y - y_j) = \varphi(x, y_j) + O(\delta^2).$$

The first term is independent of y , and so by choosing $N \sim \delta^{-2}$ to control the second term we see that $|I_j(x)| \gtrsim \delta^{n-1} \mathbb{1}_{T_j}(x)$. This allows us to estimate

$$\begin{aligned} \mathbb{E} \|T_N f\|_q &\geq \|\mathbb{E} |T_N f|\|_q && \text{by Minkowski} \\ &\gtrsim \left\| \left(\sum |I_j|^2 \right)^{1/2} \right\|_q && \text{by Khinchin (page 17)} \\ &\geq \delta^{n-1} \left\| \sum \mathbb{1}_{T_j} \right\|_{q/2}^{1/2} \end{aligned}$$

Then since $\|T_N f\|_q \lesssim \delta^{2n/q} \|f\|_p$ we can put our estimates together to obtain

$$\left\| \sum_{j=1}^M \mathbb{1}_{T_j} \right\|_{q/2} \lesssim \delta^{\frac{4n}{q} - 2(n-1)} (\delta^{n-1} M)^{2/p}$$

which is the covering lemma implying the estimate claimed. \square

An immediate corollary is that the optimal $q = \frac{2n}{n-1}$ would imply the best estimate $\|\mathcal{K}_\delta\|_{n \rightarrow n} \lesssim \delta^{-\varepsilon}$ for the curved Keakeya maximal function, and as usual this correspondence becomes very bad away from the optimal exponents. Of course, in the light of Bourgain's work we cannot conjecture that all curved Keakeya maximal functions satisfy the optimal bound, so rather we must ask ourselves a question:

Question 1. *Given a phase function, how can we find the best bound for the corresponding maximal functions? For which phases is an $L^n \rightarrow L^n$ bound of order $\delta^{-\varepsilon}$ possible?*

2.3 Maximal function bounds imply large Hausdorff dimension

Of course, there are also weaker questions to be asked about the dimension of the sets. We now prove the implication promised in the previous chapter, namely

that bounds for the maximal function imply bounds for the Hausdorff dimension of the sets. We ought to note that the tubes are not exactly the δ -neighbourhoods of the curves, having horizontal rather than perpendicular cross-section like a δ -disc. But since the direction y is bounded, the curves do not slope too much and so this makes no difference: In future we take the δ -neighbourhood of our sets to consist of tubes T_y as defined above.

Theorem 11 (Maximal function implies Hausdorff dimension). *Assume that an estimate $\|\mathcal{K}_\delta f\|_{q,\infty} \leq C\delta^{-\alpha}\|f\|_{p,1}$ holds. Then curved Kakeya sets (corresponding to phase functions of the form (2.5)) have Hausdorff dimension at least $n - p\alpha$.*

Proof: The assumption means that

$$|\{y : |\mathcal{K}_\delta \mathbb{1}_A(y)| \geq \lambda\}| \leq \left(\frac{C\delta^{-\alpha}|A|^{1/p}}{\lambda} \right)^q$$

for all $A \subseteq \mathbb{R}^n$ of finite measure, and $\lambda \in (0, 1]$. Let E be the curved Kakeya set. Recalling the definition on page 11, we have to show that $\mathcal{H}^s(E) > 0$ for all $s < n - p\alpha$. So let $E \subseteq \bigcup U_i$ where the U_i are sets with diameter less than 1. We must show that $\sum (\text{diam } U_i)^s$ is bounded below. To do this, we shall classify the U_i dyadically, and for each of the curves in E find a scale that is covering a large proportion of it, then find a single scale that is covering most of a large proportion of the curves. Then we apply \mathcal{K}_δ at this scale.

So define

$$E_k := E \cap \bigcup_{\text{diam } U_i \sim 2^{-k}} U_i$$

where the notation $r \sim 2^{-k}$ means that $2^{-k} \leq r < 2^{-(k-1)}$. For each $y \in \mathbb{B}^{n-1}$ find ω_y such that $\Gamma_y(\omega_y) \in E$. By (2.5) each curve may be parametrised as $\{x_y(t) : t \in [-1, 1]\}$ where t is the height. Now $\bigcup E_k = E$ so for each y we can find a k_y such that

$$|\{t \in [-1, 1] : x_y(t) \in E_{k_y}\}| \geq \frac{c}{k_y^2}$$

where $c = 12/\pi^2$, since the union of these sets over all k is just $[-1, 1]$. Then by similar reasoning, find a fixed K so that $K = k_y$ for all y belonging to some set $\Omega \subseteq \mathbb{B}^{n-1}$ of measure at least c/K^2 . Now define

$$\tilde{E}_K = \bigcup_{i: \text{diam } U_i \sim 2^{-K}} B_i$$

where B_i is some ball with centre in U_i and radius $2 \text{diam } U_i$. By elementary properties of balls, \tilde{E}_K is a thickened piece of E ; more precisely, \tilde{E}_K includes a

ball of radius 2^{-K} centred at each point of E_K . So for a set of t of measure at least c/K^2 the set \tilde{E}_K has such a ball centred at each $x_y(t)$ with $y \in \Omega$. Hence

$$\left| T_y^{2^{-K}}(\omega_y) \cap \tilde{E}_K \right| \geq \frac{c}{K^2} \left| T_y^{2^{-K}}(\omega_y) \right|.$$

This is where the proof in [42] uses straight lines—here we use the fact that t is the height.

We apply \mathcal{K}_δ to $f := \mathbb{1}_{\tilde{E}_K}$ with $\delta = 2^{-K}$, obtaining

$$|\{y : \mathcal{K}_{2^{-K}} \mathbb{1}_{\tilde{E}_K}(y) \geq c/K^2\}| \geq |\Omega| \geq \frac{c}{K^2}.$$

But by the restricted weak type estimate we also have

$$\begin{aligned} |\{y : \mathcal{K}_{2^{-K}} \mathbb{1}_{\tilde{E}_K}(y) \geq c/K^2\}| &\leq \left(\frac{c 2^{K\alpha} |\tilde{E}_K|^{1/p}}{c/K^2} \right)^q \\ &= C \left(K^2 2^{K\alpha} |\tilde{E}_K|^{1/p} \right)^q \end{aligned}$$

Also $|\tilde{E}_K| \leq C 2^{-nK} \#\{i : \text{diam } U_i \sim 2^{-K}\}$, so combining these and rearranging gives

$$\#\{i : \text{diam } U_i \sim 2^{-K}\} \geq C K^{-2p(1+1/q)} 2^{K(n-p\alpha)}.$$

Finally,

$$\begin{aligned} \sum_i (\text{diam } U_i)^s &\geq \#\{i : \text{diam } U_i \sim 2^{-K}\} 2^{-Ks} \\ &\geq C K^{-2p(1+1/q)} 2^{K(n-p\alpha)-Ks} \\ &\geq \text{constant} \end{aligned}$$

because $s < n - p\alpha$. So $\dim(E) \geq n - p\alpha$. \square

This shows that the best possible $L^p \rightarrow L^q$ bound (see Figure 1.3) would imply that the sets have dimension at least p , and in particular an L^n bound shows they have full dimension. But of course this is not true for all curves as Bourgain's work showed, so we ask the following:

Question 2. *For which families of curves do the associated sets have full dimension? For other curves, what is the best lower bound?*

The reader will have noticed that we proved Theorems 8 and 11 only for \mathcal{K}_δ and not \mathcal{N}_δ . Both are still true in the latter case: For Theorem 11 this requires only a minor modification of the proof, but Theorem 8 requires that we modify the phase in the Theorem as well. This is because the relationship between Kakeya and Nikodym phenomena in the curved case is a little more complicated.

For straight lines we noted that the transformation $(x', x_n) \mapsto (x'/x_n, 1/x_n)$ mapped straight lines to straight lines but exchanged the roles of position and direction. However, we are now considering curves of the form

$$\left\{ \begin{pmatrix} \omega - \nabla_y \tilde{\varphi}(t, y) \\ t \end{pmatrix} : t \in [-1, 1] \right\}$$

with $\tilde{\varphi}$ as in (2.5), which get mapped to

$$\left\{ \begin{pmatrix} s\omega - s \nabla_y \tilde{\varphi}(1/s, y) \\ s \end{pmatrix} : s \notin [-1, 1] \right\}.$$

It is easy to check that such curves correspond to phases of the form $\varphi(x, y) = \frac{1}{x_n} x' \cdot y + \tilde{\varphi}(1/x_n, y)$ in Theorem 8. If φ is linear in the first variable, then we have roughly what we found for the Bochner-Riesz problem for paraboloids on page 22. So it appears that a Kakeya problem for one type of curve might in fact correspond to a Nikodym problem for a different but related type of curve, with straight lines being a special case where the types coincide. We shall see this more concretely in the next section, where we confine our attention to a very simple class of phases.

2.4 Simple curves and phases

Notice that in both of Bourgain's theorems the bad behaviour is caused by the presence of non-linear terms in x in the phase function. For this reason, we have chosen to focus our work almost entirely on a very simple class of quadratic phases: those of the form

$$\varphi(x, y) := x' \cdot y + x_n y^\top A y + x_n^2 y^\top B y \quad (2.6)$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$ and A and B are $(n-1) \times (n-1)$ symmetric matrices over \mathbb{R} . Then (2.1) clearly holds, and (2.2) requires that $\det A \neq 0$ and in fact

$$\det(A + 2x_n B) \neq 0 \quad (2.2')$$

throughout the support of the cutoff a , which we assume to mean $x_n \in [-1, 1]$. Although very simple, these phases are general enough to exhibit many kinds of behaviour. With $B = 0$ we have the Restriction problem, while taking $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives the worst case example of Theorem 5. More interestingly still, the Generic Failure criterion of Theorem 6 has the simple form

$$B \text{ is not a multiple of } A. \quad (2.4')$$

We should note that if (2.4') does not hold, then in fact we are back in the Restriction case, since if $B = \lambda A$ we may apply the diffeomorphism

$$(x', x_n(1 + \lambda x_n)) \mapsto (x', x_n)$$

and obtain $B = 0$.

Also, by applying linear maps to x and/or y in the phase, we see that the oscillatory integral problem is invariant under congruence of the matrices. Since they are symmetric, we may assume that one of them is diagonal, or even has only 0 and ± 1 on the diagonal. This is of limited help, but in the special case where one of A, B is positive-definite we are able to simultaneously diagonalise. This will enable us to perform certain computations that seem intractable in the general case.

The simple curves have the form

$$\Gamma_y(\omega) = \left\{ \begin{pmatrix} \omega - 2tAy - 2t^2By \\ t \end{pmatrix} : t \in [-1, 1] \right\} \quad (2.7)$$

for the Kakeya problem. Now y and ω range over \mathbb{B}^{n-1} , so we may multiply either of them by an invertible matrix, and of course we can use invertible transformations of \mathbb{R}^n applied to all the curves as well. This allows us, if it is convenient, to replace A by I and B by either $A^{-1}B$ or BA^{-1} . Note however that neither of these matrices is necessarily symmetric, nor is B assumed to be invertible. By a further transformation (possibly over \mathbb{C}) we may assume that $A = I$ and that B is in Jordan normal form.

The proof of Theorem 6 is easier for quadratic phases since we do not have to take account of higher order terms, and in fact this allows us to obtain a slightly better lower bound for q .

Theorem 12 (Theorem 6 for quadratic phases). *In dimension three, if φ is of the form (2.6) with B not a multiple of A , then T_N cannot be bounded into L^q with constant $N^{-\frac{n}{q}}$ (even from L^∞) unless $q \geq \frac{28}{9} > 3$. If additionally $\text{tr}(A^{-1}B) = 0$, then this cannot hold unless $q \geq \frac{19}{6}$.*

Proof: It is enough to show that we can choose suitable $\omega = \omega(y)$ to produce a set of curves that is too small. We will use a linear function: $\omega = Wy$. We claim that it is enough to make the determinant of the map $y \mapsto x' := Wy - 2tAy - 2t^2By$ of order $|t|^m$ for small t , where $m = n = 3$ in the first case, and $m = 4$ in the second.

Fix $t \in [-\delta^{1/m}, \delta^{1/m}]$ so that the determinant is at most δ . Then if y ranges over the ball in \mathbb{R}^{n-1} of radius 1, we find that x' ranges over a set of measure at most δ , and we are interested in the size of the δ -neighbourhood of this. Now

since the eigenvalues of the map are bounded, no side of the set can be larger than $|y| < 1$, but having all sides this large would exceed the maximum permitted volume. So the worst case has $n - 2$ sides of length 1 and one thin side of length δ so that the volume does not exceed that permitted by the determinant.

Hence the largest possible neighbourhood is of measure δ . Now allowing x_n to vary over the interval $[-\delta^{1/m}, \delta^{1/m}]$ gives us that the union E of these tubes of length $\delta^{1/m}$ has measure at most $\delta^{1+1/m}$.

If the bound for T_N did hold, then by Theorem 8, \mathcal{K}_δ would satisfy

$$|\{y : \mathcal{K}_\delta \mathbb{1}_E(y) > \lambda\}| \leq \frac{C|E|^{1/r} \delta^{-2(n/r-1)}}{\lambda}$$

for all sets E and all $\lambda > 0$, where $r = \frac{q}{q-2}$. Taking E to be the set above, and $\lambda = \delta^{1/m}$ we find that the left hand side is a constant.

Hence $|E|^{1/r} \geq \delta^{2(n/r-1)+1/m}$ which implies that $1 + 1/m \leq 2n - 2r + r/m$. Rearranging gives $r \leq \frac{2n-1-1/m}{2-1/m}$. This is indeed less than the optimal $r = n$ when $m > n - 1$. In dimension three with $m = 3$ we have $r \leq 14/5$ and hence $q \geq 28/9$, while if $m = 4$ this improves to $r \leq 19/7$ and $q \geq 19/6$. This proves the claim.

So we must consider when the above condition on the determinant is satisfied. It is only here that we restrict to $n = 3$, because the calculation is more difficult for higher dimensions. By multiplying through by A^{-1} , we may take A to be the identity, with B no longer necessarily symmetric. We require $\det(W - 2tI - 2t^2B) = O(|t|^m)$. Using the multilinearity of the determinant this becomes

$$\det W - 2t \operatorname{tr} W + 2t^2(2 - (\omega_{11}b_{22} - \omega_{21}b_{12} + \omega_{22}b_{11} - \omega_{12}b_{21})) + 4t^3 \operatorname{tr} B + 4t^4 \det B.$$

Hence we need to satisfy the following equations for W :

$$\det W = 0$$

$$\operatorname{tr} W = 0$$

$$\omega_{11}b_{22} + \omega_{22}b_{11} = \omega_{21}b_{12} + \omega_{12}b_{21} + 2.$$

Clearly we could not solve these if B was a multiple of the identity. But if not, we can proceed as follows. The first two give $\omega_{22} = -\omega_{11}$ and $\omega_{12}\omega_{21} = -\omega_{11}^2$. If $b_{21} \neq 0$ then a solution is $W = \begin{pmatrix} 0 & -2/b_{21} \\ 0 & 0 \end{pmatrix}$. A similar solution exists if $b_{12} \neq 0$. Finally, if both are 0 take

$$W = \begin{pmatrix} \frac{2}{b_{22}-b_{11}} & -\frac{2}{b_{22}-b_{11}} \\ \frac{2}{b_{22}-b_{11}} & -\frac{2}{b_{22}-b_{11}} \end{pmatrix}.$$

Finally note that if we also have $\operatorname{tr} B = 0$ then we are able to take $m = 4$ rather than 3, giving the improvement stated. This completes the proof. \square

To close this chapter, we consider how these simple phases and curves behave under the transformation $(x', x_n) \mapsto (x'/x_n, 1/x_n)$, which as we previously saw, preserves straight lines but interchanges the roles of position and direction. The phase of the form (2.6) becomes

$$\varphi(x, y) = \frac{1}{x_n} x' \cdot y + \frac{1}{x_n} y^\top A y + \frac{1}{x_n^2} y^\top B y$$

and the associated curves become

$$\begin{pmatrix} t\omega - 2Ay - \frac{2}{t}By \\ t \end{pmatrix}.$$

This curve is a hyperbola, whose direction for large t is determined by ω and whose position depends on y . So a set that for each y includes one of these curves for some ω may be thought of as a Nikodym set of hyperbolas.

We can, of course, ask Kakeya questions about hyperbolas and equivalently Nikodym questions about parabolas. Although in terms of the curves this is merely swapping y and ω , it corresponds to completely different phase functions in the oscillatory integrals. These phases have the property that the set

$$\{x' : \nabla_y \varphi(x, y) = \omega\} = \{y - 2x_n A \omega - 2x_n^2 B \omega\}$$

for the Nikodym question about parabolas, or

$$\{x' : \nabla_y \varphi(x, y) = \omega\} = \left\{ x_n y - 2A\omega - \frac{2}{x_n} B\omega \right\}$$

for the Kakeya problem with hyperbolas. These two are equivalent and the two phases related via the usual transformation. Solving these equations we obtain the phases

$$\varphi(x, y) = \frac{1}{4x_n} y^\top (A + x_n B)^{-1} y - \frac{1}{2x_n} y^\top (A + x_n B)^{-1} x' \quad (2.8)$$

$$\varphi(x, y) = \frac{1}{4} x_n y^\top \left(A + \frac{1}{x_n} B \right)^{-1} y - \frac{1}{2} y^\top \left(A + \frac{1}{x_n} B \right)^{-1} x' \quad (2.9)$$

respectively. Later on we shall see that for certain matrices A, B , these phases behave better than their simpler-looking counterparts defined earlier.

2.5 Summary

Let us pause to summarise the background material presented so far, and to outline the structure of the rest of the thesis.

In these two chapters we have seen several instances of a chain of implications, namely that oscillatory integral estimates imply estimates for maximal

functions, which imply lower bounds for the Hausdorff dimension, and in turn for the Minkowski dimension, of related sets. For the classical Restriction and Bochner-Riesz problems, the maximal functions and sets involved straight lines, while for the more general oscillatory integrals studied by Hörmander, curves arose. The phase function in the integral determined the shape of the curves appearing in the maximal functions and sets: Precisely, a phase φ gives rise to curves $\{x : \nabla_y \varphi(x, y) = \omega\}$.

We have also seen how the transformation $(x, x') \mapsto (x'/x_n, 1/x_n)$ gives rise to equivalent problems that look geometrically different. Straight lines are preserved under this transformation, while parabolas become hyperbolas and vice versa. Also, the roles of “position” and “direction” are roughly interchanged, so that a Kakeya-type problem becomes a Nikodym-type problem. This information is summarised in Figure 2.2. Note that in the Figure, our use of the notations y and ω has been kept consistent: The maximal function is always a function of y , and our sets always include a curve for each y (with $\omega = \omega(y)$). The problem is described as “Kakeya-type” if y governs the direction of the curve and ω the position, and “Nikodym-type” if the reverse is true. However, in the rest of the thesis, we shall prefer to always use parabolas $\Gamma_y(\omega)$ of the form (2.7), and denote the argument of the Nikodym maximal function by ω as we originally defined it on page 12.

We have also seen (Theorem 12) that the results hoped for in the straight line case cannot be true for curves in general, even where φ is merely quadratic rather than linear in x_n . So our intention is to find out what properties of φ cause good or bad behaviour, by attempting to apply the techniques used in the straight line case to curves.

First of all we check that the questions about dimension are sensible, by showing that the sets under consideration can indeed have zero measure.

Then in Chapter 4 we obtain the easiest and broadest result: an $L^{\frac{n+1}{2}}$ bound for the maximal function for every family of curves. We also discuss the case $n = 2$, and the geometric meaning of the basic assumptions on φ .

Then we turn our attention to the bounds that are achieved by geometric methods. Using a method due to Katz, we succeed in proving a lower bound of $\frac{n+2}{2}$ for the dimension of Nikodym sets of parabolas with $BA^{-1}B = 0$, and discuss the difficulty of achieving this for other curves. We also attempt to apply the approaches of Wolff and Bourgain to the curved case, and show where the methods break down.

Finally in Chapter 6 we consider the newer arithmetic methods of Bourgain, Katz and Tao. These give rise to combinatorial problems which we can only

<p>Restriction</p> $\varphi(x, y) = x' \cdot y + x_n y^\top y$ <p>Bochner-Riesz</p> $\varphi(x, y) = \frac{1}{x_n}(x' \cdot y + y^\top y)$	\Rightarrow	<p>Keakeya (Straight Lines)</p> $\begin{pmatrix} \omega - 2ty \\ t \end{pmatrix}$ \Leftrightarrow <p>Nikodym (Straight Lines)</p> $\begin{pmatrix} t\omega - 2y \\ t \end{pmatrix}$
<p>Hörmander</p> $\varphi(x, y) = x' \cdot y + x_n y^\top Ay + x_n^2 y^\top By$ <p>Hörmander</p> $\varphi(x, y) = \frac{1}{x_n} x' \cdot y + \frac{1}{x_n} y^\top Ay + \frac{1}{x_n^2} y^\top By$	\Rightarrow \Leftrightarrow \Rightarrow	<p>Keakeya (Parabolas)</p> $\begin{pmatrix} \omega - 2tAy - 2t^2 By \\ t \end{pmatrix}$ \Leftrightarrow <p>Nikodym (Hyperbolas)</p> $\begin{pmatrix} t\omega - 2Ay - \frac{2}{t} By \\ t \end{pmatrix}$
<p>Hörmander</p> $\varphi(x, y) = \frac{1}{4} x_n y^\top (A + \frac{1}{x_n} B)^{-1} y - \frac{1}{2} y^\top (A + \frac{1}{x_n} B)^{-1} x'$ <p>Hörmander</p> $\varphi(x, y) = \frac{1}{4x_n} y^\top (A + x_n B)^{-1} y - \frac{1}{2x_n} y^\top (A + x_n B)^{-1} x'$	\Rightarrow \Leftrightarrow \Rightarrow	<p>Keakeya (Hyperbolas)</p> $\begin{pmatrix} ty - 2A\omega - \frac{2}{t} B\omega \\ t \end{pmatrix}$ \Leftrightarrow <p>Nikodym (Parabolas)</p> $\begin{pmatrix} y - 2tA\omega - 2t^2 B\omega \\ t \end{pmatrix}$

Figure 2.2: Relationships between Oscillatory integrals and Keakeya & Nikodym problems for parabolas and hyperbolas

partially solve. We obtain bounds for the same Nikodym sets as in Chapter 5, and for certain Keakeya sets, which greatly exceed $\frac{n+2}{2}$ in high dimensions.

Chapter 3

Sets can have measure zero

*It's OK to divide by zero, provided it's on a set of zero measure.
Tony Carbery*

3.1 Introduction

The bulk of this thesis is devoted to the largeness of curved Kakeya sets, in terms of their dimension. But before considering that, we ought to check that such sets can be small enough for the question of their dimension to become interesting—that is, do there exist examples with (Lebesgue) measure zero?

In this chapter we show that the answer is yes for both the Kakeya- and Nikodym-type sets under consideration. In fact, our result will be a quite general one showing that every small enough family of smooth curves or surfaces can be made to lie in a null set provided that one is free to control enough of the parameters. There are several reasons for adopting this more general approach.

- It allows us to obtain the Kakeya and Nikodym results together.
- We do not have to use special properties of the phase function φ so we are not restricted to quadratic, nor polynomial, phases.
- Unlike in the straight line case, it is not enough solely to prove the existence of such sets in the plane, at least not for general curves. This is because a cylinder produced from a curved Kakeya set in the plane need not include all the curves we want in higher dimensions. For the particular parabolas we are looking at, it happens to work in certain circumstances as proved below, but this is more of an algebraic accident.
- The traditional proofs used in the straight line case do not generalise readily to curves. Those that involve cutting up a simple shape and moving the

pieces, while very appealing, look unlikely for parabolas and hopeless in general. Kahane's approach via the Cantor set also seems difficult because of the non-linearities involved.

We now briefly show why Keakeya sets of certain parabolas are easier than general curves:

Proposition 13. *Given a Keakeya set of curves of the form (2.7) where the matrix $A^{-1}B$ has at least one real eigenvalue, we can apply a diffeomorphism so that the resulting set has a projection consisting only of straight lines.*

Proof: As discussed previously (page 32) we may replace A by I and B by $A^{-1}B$. Also, the question of the measure of these sets is invariant under similarity. Now a matrix with a real eigenvalue λ is similar over the reals to a matrix whose last row consists of zeroes except for the diagonal entry, which is λ . With the matrix in this form, the $n - 1$ th component of the curves in question is $\omega_{n-1} - 2ty_{n-1} - 2\lambda t^2 y_{n-1}$ while the n th component is just t as always. Applying the transformation $t + \lambda t^2 \mapsto t$, which is a diffeomorphism on $|t| < 1$ by Hörmander's condition (2.2') on page 31, straightens out the projection of the curves onto the x_{n-1}, x_n -plane. \square

Hence the straight line results are enough to show measure zero for very special curved Keakeya sets, but if all the eigenvalues are complex, or indeed if the curves are of a more complicated form, then we must find some other way to proceed.

So, how can we prove a Keakeya or Nikodym result for general curves? The main idea is due to Sawyer [30], who showed the following:

Theorem 14. *There is a function ψ on \mathbb{R} such that whenever g is a real-valued Borel measurable function on (a subset of) $\mathbb{R} \times \mathbb{R}^{n-1}$ with the property that $y \mapsto g(y, t)$ is C^1 for a.e. t , the set*

$$E_f := \bigcup_y \{(x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} : x = g(y, t) - \psi(y)\}$$

has measure zero.

That is, a smooth one-parameter family of measurable hypersurfaces may be translated to lie in a null set. Moreover, the translations may be taken parallel to \mathbb{R} and need not depend on g .

For our purposes we need to generalise this to higher codimension, since curves of course have codimension $n - 1$. Also, we do not want to be restricted to using translations, since in the Nikodym case it is the directions we are allowed to vary while the positions are kept fixed. So we remove all distinction between

“shape parameters” and “position parameters”, simply denoting those that are “given” by y and those we are free to choose by ω , in accordance with the usage in Figure 2.2.

Consider d -dimensional objects of the following form

$$\Gamma(y, \omega) := \left\{ \begin{pmatrix} f(y, \omega, t) \\ t \end{pmatrix} : t \in \mathbb{R}^d \right\}$$

where $f : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$. So $\Gamma(y, \omega)$ is a d -dimensional surface in \mathbb{R}^n , and the family of them has $p + q$ parameters in total. In the case above, $f(y, \omega, t) = g(y, t) - \omega$.

Our aim is to show that under certain hypotheses, a null set may include a representative of every combination of the first p parameters provided that the remaining q parameters can be chosen to depend on them. That is, there exists a set of measure zero that includes a $\Gamma(y, \omega(y))$ for every y . In fact, this function of y will be the obvious generalisation of Sawyer’s universal translation function ψ , and will not depend on f .

More precisely, our theorem is the following

Theorem 15. *There is a function $\psi : \mathbb{R}^p \rightarrow \mathbb{R}^q$ with the following property: Let $f : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$ where $p \leq n - d \leq q$ and $d < n$. Suppose that f is measurable, that for fixed t the map $(y, \omega) \mapsto f(y, \omega, t)$ is C^1 , that the Jacobian $\frac{\partial f}{\partial \omega}$ always has full rank (namely $n - d$) and that both Jacobians are Lipschitz. Then the set*

$$E_f := \bigcup_{y \in \mathbb{R}^p} \Gamma(y, \psi(y))$$

has measure zero.

The proof will have three parts. First, we define the universal transformation function ψ , which will be the obvious higher dimensional analogue of that used by Sawyer. Then, we show that all of the slices through the set at fixed t have zero measure, which is where the conditions on the Jacobians are used. Finally we show that the whole set is measurable, using the C^1 condition. This allows us to apply Fubini’s theorem to obtain the result.

3.2 Definition of ψ

We begin with a few easily verified facts needed for the proof.

(i) Factorial Expansion: Every $a \in (0, 1]$ has a unique expansion of the form

$$a = \sum_{n=2}^{\infty} \frac{a_n}{n!}$$

where the a_n are integers, $0 \leq a_n \leq n - 1$ and infinitely many of the coefficients are non-zero.

(ii) There are countably many numbers in $(0, 1]$ that also have a finite factorial expansion. Call these numbers *bad*.

$$(iii) \sum_N^{\infty} \frac{n-1}{n!} = \frac{1}{(N-1)!}$$

All norms, whether of matrices or vectors, will denote the largest absolute value of the entries—this is merely to avoid keeping track of constants, since of course all norms on a finite dimensional space are equivalent.

We shall use subscripts to denote the coefficients of the factorial expansions of the vectors y rather than their components. Thus for $y \in (0, 1]^p$ we can write $y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$ in the natural way.

Our aim is to construct a kind of “universal transformation function” $\psi : \mathbb{R}^p \rightarrow \mathbb{R}^q$ by generalising the approach in [30]. The plan is that ψ will be a series similar to the factorial expansion of y , and we hope to make $f(y, \psi(y), t)$ close to that value of f where the series for both of the first two arguments are truncated—a finite set of values. We choose the coefficients in the series to get rid of the main error term; it turns out that the coefficients therefore must correspond to the values of $\frac{\partial f}{\partial \omega}^{-1} \frac{\partial f}{\partial y}$. So we need to devise a sequence of $q \times p$ real matrices that is in some sense ‘dense’ and takes on arbitrarily large values, but does not grow too quickly. This is what we shall do now.

For $k \geq 3$ set

$$D_k = \{(y_2, \dots, y_{k-1}) \in {}^{(k-2)}\mathbb{Z}^p : 0 \leq y_n^i \leq n-1\},$$

where the notation means a set of $(k-2)$ -tuples of those p -dimensional vectors that can form the first $k-2$ coefficients in a factorial expansion. Let Ω_k be the set of all maps $D_k \rightarrow {}^q[-\log \log \log k, \log \log \log k]^p$, that is, $q \times p$ matrices whose elements are bounded by $\log \log \log k$. Next let $\{s_j^k\}_{j=1}^{m_k}$ be a finite $1/k$ -dense subset of Ω_k , meaning that

$$\forall_{s \in \Omega_k} \exists_j \forall_{(y_2, \dots, y_{k-1}) \in D_k} \|s(y_2, \dots, y_{k-1}) - s_j^k(y_2, \dots, y_{k-1})\| < \frac{1}{k}.$$

At this point it will be helpful to notice that $m_k \sim (k \log \log \log k)^{pq(k-1)!^p}$, by taking the number of possible matrices and raising it to the power of the number of arguments in the function.

Next we define the sequence of maps to use as coefficients in the definition of ψ . Call $r \in \Omega_l$ an *extension* of $s \in \Omega_k$ if $l \geq k$ and for all $(y_2, \dots, y_{l-1}) \in D_l$

we have $r(y_2, \dots, y_{l-1}) = s(y_2, \dots, y_{k-1})$. Set $r_2 \equiv 1$ and for each $n \geq 3$ choose $r_n \in \Omega_n$ so that for all $k \geq 3$ and $1 \leq j \leq m_k$ there is an r_n that is an extension of s_j^k .

Now for $y \in (0, 1]^p$ define ψ by

$$\psi(y) = \sum_{n=2}^{\infty} r_n(y_2, \dots, y_{n-1}) \frac{y_n}{n!}$$

where each summand contains a matrix multiplication. Finally, extend ψ to all of \mathbb{R}^p by periodicity.

We shall need some continuity properties of ψ .

Lemma 16. *Suppose that y and \bar{y} have the same factorial expansion up to the N th term (meaning $y_N, N \geq 2$). Then $|y - \bar{y}| \leq 1/N!$ and $|\psi(y) - \psi(\bar{y})| \leq C \log \log \log N/N!$.*

Proof: $|y - \bar{y}| \leq \sum_{n=N+1}^{\infty} \frac{|y_n - \bar{y}_n|}{n!} \leq \sum_{n=N+1}^{\infty} \frac{n-1}{n!} = 1/N!$. Similarly

$$\begin{aligned} |\psi(y) - \psi(\bar{y})| &\leq \sum_{n=N+1}^{\infty} \frac{(n-1) \log \log \log n}{n!} \\ &\leq \frac{N \log \log \log(N+1)}{(N+1)!} + \sum_{n=N+2}^{\infty} \frac{(n-1) \log \log \log n}{n!} \\ &\leq \frac{C \log \log \log N}{N!} + C \sum_{n=N+2}^{\infty} \frac{n-2}{(n-1)!} \\ &= \frac{C \log \log \log N}{N!} \end{aligned} \quad \square$$

In particular, this shows that ψ is continuous except at points where one of the components can also have a terminating factorial expansion. At such points there is left continuity in the “bad components” and the right limits exist.

3.3 Slices have measure zero

We now need to show that for suitable values of n, d, p, q this ψ has the property claimed, that is, the set

$$E_f := \bigcup_y \Gamma(y, \psi(y))$$

has measure zero. In this section we show that almost all of the slices through the set at fixed t have measure zero; since t is fixed we suppress it and just prove the following:

Lemma 17. *Let $f : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^{n-d}$, with $p \leq n - d \leq q$ and $d < n$. Then if f is C^1 and $\frac{\partial f}{\partial \omega}$ always has highest possible rank (namely $n - d$) and both Jacobians of f are Lipschitz, then the range of $f(\cdot, \psi(\cdot))$ is of measure zero.*

These hypotheses are very natural: $d < n$ is merely to avoid trying to pack n -dimensional objects in \mathbb{R}^n , and the other inequalities mean that we should not try to include too large a family of surfaces, and we must be free to choose many of the parameters. The condition about the rank simply says that the surface must actually depend on the parameters that we are free to vary.

Proof: By periodicity it is enough to consider only $(0, 1]^p$. For a vector y and natural number k write

$$y^{(k)} = \sum_{n=2}^{k-1} \frac{y_n}{n!} \quad \psi^{(k)}(y) = \sum_{n=2}^{k-1} r_n(y_2, \dots, y_{n-1}) \frac{y_n}{n!}.$$

Then for all natural numbers k and N we have

$$\begin{aligned} f(y, \psi(y)) &= \left[f(y, \psi(y)) - f(y^{(N)}, \psi(y)) - \frac{\partial f}{\partial y}(y^{(k)}, \psi^{(k)}(y))(y - y^{(N)}) \right] \\ &+ \left[f(y^{(N)}, \psi(y)) - f(y^{(N)}, \psi^{(N)}(y)) - \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y))(\psi(y) - \psi^{(N)}(y)) \right] \\ &+ \left[\frac{\partial f}{\partial y}(y^{(k)}, \psi^{(k)}(y)) + \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y))r_N(y_2, \dots, y_{N-1}) \right] \frac{y_N}{N!} \\ &+ \sum_{n=N+1}^{\infty} \left[\frac{\partial f}{\partial y}(y^{(k)}, \psi^{(k)}(y)) + \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y))r_n(y_2, \dots, y_{n-1}) \right] \frac{y_n}{n!} \\ &+ f(y^{(N)}, \psi^{(N)}(y)) \\ &=: I(y) + II(y) + III(y) + IV(y) + V(y). \end{aligned}$$

The final term takes a very large, but finite, number of values, so our task is to show that the other terms are correspondingly extremely small.

Let $\varepsilon > 0$ be given. Using the hypothesis that f is C^1 , choose k just large enough that the following hold:

- (i) If both $|y - \bar{y}| < \frac{1}{(k-1)!}$ and $|\omega - \bar{\omega}| < \frac{\log \log \log k}{(k-1)!}$, then $\left\| \frac{\partial f}{\partial y}(y, \omega) - \frac{\partial f}{\partial y}(\bar{y}, \bar{\omega}) \right\| < \varepsilon$ and $\left\| \frac{\partial f}{\partial \omega}(y, \omega) - \frac{\partial f}{\partial \omega}(\bar{y}, \bar{\omega}) \right\| < \varepsilon$.
- (ii) $\left\| \frac{\partial f}{\partial y}(y, \omega) \right\| < \log \log \log k$ and $\left\| \frac{\partial f}{\partial \omega}(y, \omega) \right\| < \log \log \log k$.
- (iii) $\left\| \frac{\partial f}{\partial \omega}^{-1} \frac{\partial f}{\partial y}(y, \omega) \right\| < \log \log \log k$ where $\frac{\partial f}{\partial \omega}^{-1}$ is a right inverse of the $(n-d) \times q$ matrix $\frac{\partial f}{\partial \omega}$. Here we are using the assumption that $q \geq n-d$.
- (iv) $\frac{\log \log \log k}{k} < \varepsilon$

It is necessary to estimate k in terms of ε . First note that conditions (ii) and (iv) do not play a role in this since these quantities are bounded independently of ε . By the Lipschitz assumption (which is actually far stronger than needed)

we see that (i) is a weak requirement on k , so we conclude that (iv) is sharp: $\frac{\log \log \log k}{k} \approx \varepsilon$ and so certainly $k \lesssim 1/\varepsilon^2$.

Next, find an s_j^k within $1/k$ of the matrix $-\frac{\partial f}{\partial \omega}^{-1} \frac{\partial f}{\partial y}(y^{(k)}, \psi^{(k)}(y))$. Then find N such that r_N is an extension of s_j^k . We show that parts I–IV above are smaller than $\frac{\varepsilon \log 1/\varepsilon}{(N-1)!}$.

Part I is handled using the mean value theorem. The i th component of $I(y)$ is

$$f^i(y, \psi(y)) - f^i(y^{(N)}, \psi(y)) - \nabla_y f^i(y^{(k)}, \psi^{(k)}(y)) \cdot (y - y^{(N)})$$

which, by the one-dimensional mean value theorem in the direction $y - y^{(N)}$, equals $(\nabla f^i(\xi, \psi(y)) - \nabla f^i(y^{(k)}, \psi^{(k)}(y))) \cdot (y - y^{(N)})$ for some $\xi \in [y^{(N)}, y]$. But then $|\xi - y^{(k)}| < \frac{1}{(k-1)!}$, and $|\psi(y) - \psi^{(k)}(y)| < \frac{\log \log \log k}{(k-1)!}$ so that by applying (i) to this and all the other components we eventually get

$$\begin{aligned} |I(y)| &\leq \varepsilon |y - y^{(N)}| \\ &\leq \varepsilon \sum_N \frac{1}{n!} |y_n| \\ &\leq \varepsilon \frac{1}{(N-1)!}. \end{aligned}$$

II works similarly, except that we end up with

$$|II(y)| \leq \varepsilon |\psi(y) - \psi^{(N)}(y)| \leq C\varepsilon \frac{\log \log \log N}{(N-1)!}.$$

But note that N was chosen to make r_N an extension of s_j^k , so that provided we ordered the sequence (r_n) sensibly, we have

$$\begin{aligned} N &\leq \sum_{l < k} m_l + j \\ &\leq Ck(k \log \log \log k)^{pq(k-1)!p}. \end{aligned}$$

By a wasteful application of our estimate for k , we then have $\log \log \log N \lesssim \log \frac{1}{\varepsilon}$. So the estimate of $C \frac{\varepsilon \log(1/\varepsilon)}{(N-1)!}$ for II follows. (This is the step for which we need the rather unlikely-looking triple log—in Sawyer’s proof this issue does not arise, because there $\frac{\partial f}{\partial \omega} = -I$ and so this term cancels with parts of III and IV.)

For III, our choice of N gives us cancellation.

$$\begin{aligned} |III(y)| &\leq 0 + \frac{1}{k} \left| \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y)) \right| \frac{|y_N|}{N!} \\ &\leq \frac{\log \log \log k}{k} \frac{N-1}{N!} \\ &< \frac{\varepsilon}{(N-1)!}. \end{aligned}$$

Finally,

$$\begin{aligned}
IV(y) &\leq \sum_{n=N+1}^{\infty} \frac{\log \log \log k (1 + \log \log \log n)(n-1)}{n!} \\
&\leq \log \log \log k \left[\frac{CN \log \log \log N}{(N+1)!} + \sum_{N+2} \frac{(n-1) \log \log \log n}{n!} \right] \\
&\leq \frac{C \log \log \log k \log \log \log N}{N!} \\
&< \frac{C\varepsilon}{(N-1)!}.
\end{aligned}$$

Combining these estimates we see that

$$\text{range}(f(\cdot, \psi(\cdot))) \subseteq \bigcup_{z \in \text{range}(V)} \text{Ball}(z, \frac{C\varepsilon \log 1/\varepsilon}{(N-1)!}).$$

But $V(y)$ depends only on y_2, \dots, y_{N-1} , so $\text{range}(V)$ has at most $(N-1)^p$ elements. Hence

$$\begin{aligned}
|\text{range}(f(\cdot, \psi(\cdot)))| &\leq (N-1)^p \left(\frac{C\varepsilon \log 1/\varepsilon}{(N-1)!} \right)^{n-d} \\
&\leq C \frac{(\varepsilon \log 1/\varepsilon)^{n-d}}{(N-1)!^{n-d-p}}
\end{aligned}$$

which, since ε is arbitrary, proves the result since $p \leq n-d$ and $d < n$. \square

3.4 The whole set is measurable

To conclude the proof of the theorem we must show that the entire set E_f is measurable. Although this is hardly a surprising fact, the proof will unfortunately be rather technical. The main idea is that E_f differs from a measurable set by a set of measure zero, and the difference is caused by the discontinuities of ψ at bad values of y . In Sawyer's case there were only countably many such values, but here in higher codimensions this is not true since y need only have one bad component to produce a discontinuity, so instead we must show that the difference set is null by showing that it may be included in a set of arbitrarily small measure.

Lemma 18. *Let $f : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^d \rightarrow \mathbb{R}^{n-d}$ where $p \leq n-d \leq q$ and $d < n$. Suppose that f is measurable, and is C^1 in (y, ω) for all t . Then there exists a measurable set B and a null set E' such that*

$$E_f \subseteq B \subseteq E_f \cup E'.$$

Proof: Write $\mathbb{R}^p \times \mathbb{R}^q = \bigcup_{n=1}^{\infty} D_n$ where each D_n is open with compact closure. Let

$$B(y, n, k) = \left\{ (x, t) \in \mathbb{R}^{n-d} \times \mathbb{R}^d : |f(y, \psi(y), t) - x| < \frac{1}{k}, (y, t) \in D_n \right\}$$

be the set of points near to the surface $\Gamma(y, \psi(y))$. Define

$$B := \bigcup_{n=1}^{\infty} \bigcap_{k=2}^{\infty} \bigcup_{y \in \mathbb{Q}^p} B(y, n, k)$$

so that a point belongs to B if and only if there are curves arbitrarily close to it whose “direction” y is from \mathbb{Q}^p . Certainly $E_f \subseteq B$, and B is a measurable set, so it remains to check that we have not included too much extra. Because f is continuous, the only extra points in B come from the discontinuities of ψ . That is, we have added all surfaces of the form

$$\Gamma\left(y, \lim_{z \xrightarrow[S]{} y} \psi(z)\right)$$

where $S \subseteq [p]$ and the notation $\xrightarrow[S]{} y$ means that we take right limits for those components in S and left limits otherwise. This gives a different surface only when the components in S are bad. (Recall that bad numbers are those that can have a terminating factorial expansion.)

Let E_S denote the union of the extra curves obtained in this way when y^j is bad for all $j \in S$. We must show that E_S is a null set.

Clearly if $S = [p]$ then since the set of bad numbers is countable, so is the set of bad vectors y , so $E_{[p]}$ is just a countable union of d -dimensional surfaces, which certainly has zero measure.

If S is not the whole of $[p]$ then there are uncountably many vectors y to consider, since the other components need not be bad. We shall show however that if we fatten the surfaces slightly, finitely many y will be sufficient to cover the rest.

To simplify notation we do the case $S = \{1\}$; the modification for other sets will then be obvious.

$$E_{\{1\}} = \bigcup_{b \text{ bad}} \left\{ \Gamma\left(y, \lim_{z \xrightarrow[\{1\}]{} y} \psi(z)\right) : y = (b, y_2, \dots, y_p) \text{ some } y_2, \dots, y_p \in \mathbb{R} \right\}$$

Denote the sets on the right hand side by $E_{\{1\}}(b)$. There are countably many b , so it is enough to show that each $E_{\{1\}}(b)$ has zero measure.

Suppose that we have another vector $\bar{y} = (b, \bar{y}_2, \dots, \bar{y}_p)$ and that the components of y and \bar{y} have the same factorial expansion up to the N th term. By Lemma 16 we have

$$|y - \bar{y}| \leq \frac{1}{N!} \quad \text{and} \quad |\psi(z) - \psi(\bar{z})| \leq \frac{C \log \log \log N}{N!}.$$

Restricting to a bounded set we then have

$$\left| f(y, \lim_{z \xrightarrow{S} y} \psi(z), t) - f(\bar{y}, \lim_{z \xrightarrow{S} \bar{y}} \psi(z), t) \right| \leq \frac{C \log \log \log N}{N!}$$

because the derivatives of f are continuous and therefore bounded on bounded sets. Since y is bounded, counting the number of possibilities for the first N terms of its expansion we find that $E_{\{1\}}(b)$ is included in at most $(N-1)!^{p-\#S}$ sets of the form

$$\left\{ (x, t) \in \mathbb{R}^{n-d} \times \mathbb{R}^d : \left| f(y, \lim_{z \xrightarrow{S} y} \psi(z), t) - x \right| \leq \frac{C \log \log \log N}{N!} \right\},$$

which gives

$$|E_{\{1\}}(b)| \leq \left(\frac{C \log \log \log N}{N!} \right)^{n-d} (N-1)!^{p-\#S} = \frac{C (\log \log \log N)^{n-d}}{N^{n-d} (N-1)!^{n-d-p+\#S}}.$$

Hence, since $n \geq p + d$ and N is arbitrary, $|E_{\{1\}}(b)| = 0$. By countable unions we can remove the restriction to a bounded set. The above formulae hold for more general sets S . So finally letting $E' = \cup_{S \subseteq [p]} E_S$ we have $|E'| = 0$ and $B \subseteq E_f \cup E'$ as required. \square

3.5 Discussion

These lemmas together with Fubini's theorem complete the proof of Theorem 15. We remark that our hypotheses are stronger than needed: Those only needed in Lemma 17 could be restricted to a.e. t , and the hypotheses of C^1 for every t in Lemma 18 could be weakened to a Lipschitz condition. The Lipschitz condition on the Jacobians was only used to show that condition (iv) on page 42 is more stringent than (i): this would still hold with a weaker condition on the modulus of continuity of the Jacobians. In fact, we can do without any such condition if we sacrifice the universality of ψ and allow it to depend on the rate of growth of the derivatives of f . But since it is not known how much smoothness is really required (although some is needed as Sawyer shows) we do not pursue this here, preferring to keep the hypotheses simple.

The theorem is certainly sufficient for our purposes. If our phase function φ is of the form $\varphi(x, y) = x' \cdot y + \tilde{\varphi}(x_n, y)$, then the curves we must consider, namely the solutions of $\nabla_y \varphi = \omega$, have the required form for the theorem, with $f(y, \omega, t) := \omega - \nabla_y \tilde{\varphi}(t, y)$. The hypotheses are all satisfied, since our phases are assumed to be infinitely smooth, and here $d = 1$ and $p = q = n - 1$. Similarly the corresponding Nikodym-type problems give $f(y, \omega, t) := t\omega - t \nabla_y \tilde{\varphi}(1/t, y)$ which again satisfies the hypotheses, since we still have smoothness apart from at $t = 0$, but this value is excluded from our definition of the Nikodym set.

Corollary 19. *Curved Kakeya or Nikodym sets in all dimensions can have measure zero.*

This was our main aim. But in fact our theorem sheds some light on other known results on curve-packing. For example, a null set in the plane can be constructed so as to include a circle of every radius (Besicovitch and Rado, Kinney 1968), but if a set has a circle centred at every point in the plane then it must have positive measure (Bourgain 1986, Marstrand 1987). However, with circles centred at all points on a curve the set can still be null (Talagrand 1980). These examples illustrate the numerology of the theorem and suggest that the conditions on the parameters might in fact be necessary as well as sufficient. Higher dimensional examples include the k -plane problem: A set in \mathbb{R}^3 that includes a plane in every direction must have positive measure (Marstrand 1979, Falconer 1980)—what can be said about packing k -planes in \mathbb{R}^n ? This problem has been studied by Falconer, Bourgain and others but remains unsolved. If the numerology of Theorem 15 was found to be sharp, then k -planes could be packed into a null set only when $k = 1$. References for these and similar results can be found in [14], [25] and [42].

Chapter 4

“Trivial” Bounds

This is where all the depth lies, and as you will see, it is trivial.
Tony Carbery

4.1 The plane case

The earliest result on the dimension of Kakeya sets was that of Davies [13], who in 1971 showed that plane Kakeya sets must have dimension two. His proof used point-line duality, but in the same year Córdoba [12] proved the result for the Nikodym maximal function using an appealing geometric argument. The main idea was that if two $1 \times \delta$ rectangles have centres separated by d , then the area of their intersection is at most $C\delta/d$. This simple observation is true for curved tubes also, and its higher-dimensional version will be proved in the next section.

However, in the plane the curved case really reduces to the case of straight lines. In fact, when $n = 2$ our matrices A and B are simply numbers, so we cannot help but have B a multiple of A , and as shown on page 32 this reduces by a change of variable to the straight line case.

Of course, if we allow higher order terms in the phase function, then the reduction is not so simple. However, Tao has shown [38] that all Hörmander-type oscillatory integral problems in two dimensions can be reduced to the two dimensional Restriction problem. Therefore the curved case becomes interesting only in dimensions three and above.

4.2 The bound $\frac{n+1}{2}$

The $L^{\frac{n+1}{2}}$ bound for the Nikodym maximal function for straight lines was first proved in 1986 by Christ, Duoandikoetxea and Rubio de Francia [11] using Fourier transform methods. Since then, more geometric proofs have been given. One of

these, appropriate for the set dimension question, holds the beginning of the arithmetic methods which we look at in Chapter 6. Here however, we use the “bush argument” of Bourgain [2], in which the main idea is the estimate for the size of the intersection of two different tubes.

First we look at the way curved tubes can intersect, and prove the $L^{\frac{n+1}{2}}$ bound for all of the curves we have been considering. We then show geometrically why the non-degeneracy criterion (2.2) is crucial, by providing a counterexample to the maximal function result if it is not assumed. We also prove the slightly curious fact that even without non-degeneracy, the Minkowski result still holds.

4.2.1 Maximal function result for non-degenerate curves

We begin with the promised lemma on intersection. Describe two tubes $T_y(\omega)$ and $T_{\bar{y}}(\bar{\omega})$ as d -separated if $|y - \bar{y}| \geq d$ (when considering the Keakeya problem) or if $|\omega - \bar{\omega}| \geq d$ (when considering the Nikodym problem).

Lemma 20. *Assuming (2.2), there is a constant C depending only on A and B such that if two d -separated δ -tubes meet, then the diameter of their intersection is at most $C\delta/d$.*

Proof: By the linearity of (2.7) in y and ω it is enough to show that if $\Gamma_y(\omega)$ and $\Gamma_0(0)$ are d -separated and meet at height t_0 , then $|\omega - 2tAy - 2t^2By| > 2\delta$ for all t that are at least $C\delta/d$ away from t_0 . The fact that the curves meet means that $\omega = 2t_0Ay + 2t_0^2By$.

(i) Suppose that $|y| \geq d$.

$$\begin{aligned} |\omega - 2tAy - 2t^2By| &= 2|t_0 - t| |(A + (t_0 + t)B)y| \\ &\geq 2|t_0 - t||y| \min\{|\lambda| : \lambda \text{ an eigenvalue of } A + (t_0 + t)B\} \\ &> 2|t_0 - t|d \min\{|\lambda| : \lambda \text{ an eigenvalue of } A + 2sB, s \in [-1, 1]\} \end{aligned}$$

Now we know from (2.2) that $\det(A + 2sB) \neq 0$ for all $s \in [-1, 1]$. We claim that for all s there exist $\varepsilon_s > 0$, $\delta_s > 0$ such that whenever t is within δ_s of s the eigenvalues of $A + tB$ are all at least ε_s in modulus. This is true because a small variation in t corresponds to a small variation in the coefficients of the characteristic polynomial of $A + tB$. In fact since A and B are symmetric the eigenvalues remain real as t is varied, so the size of the smallest eigenvalue is even a continuous function.

Then by an obvious compactness argument the minimum occurring above is non-zero, and we let C be its reciprocal to obtain $|\omega - 2tAy - 2t^2By| > 2\delta$ whenever $|t_0 - t| > \frac{C\delta}{d}$, as required.

(ii) Suppose that $|\omega| \geq d$. Then since $\omega = 2t_0Ay + 2t_0^2By$ and the eigenvalues of this matrix are bounded above, it follows that $|y| \geq cd$. So the result follows from the first part. \square

We are now able to prove the $L^{\frac{n+1}{2}}$ bound for \mathcal{K}_δ and \mathcal{N}_δ

Theorem 21. *Assuming (2.2), the curved Keakeya maximal function \mathcal{K}_δ satisfies*

$$\|\mathcal{K}_\delta f\|_q \leq C_\varepsilon \delta^{-(n/p-1+\varepsilon)} \|f\|_p \quad (4.1)$$

for $1 \leq p \leq \frac{n+1}{2}$ and $1 \leq q \leq (n-1)p'$.

Proof: It is enough to show the relevant restricted weak type estimate at the endpoint, namely

$$|\{y : \mathcal{K}_\delta \mathbb{1}_E(y) > \lambda\}| \lesssim \left(\frac{|E|^{\frac{2}{n+1}}}{\delta^{\frac{2n}{n+1}-1}\lambda} \right)^{n+1} = \frac{|E|^2}{\delta^{n-1}\lambda^{n+1}}$$

where $\mathbb{1}_E$ is the characteristic function of the set E . So let $E \subseteq \mathbb{R}^n$ and λ be given and denote the set on the left hand side by Ω . Pick a maximal d -separated subset $\{y_j\}_{j=1}^M \subset \Omega$, where we choose $d = \frac{10C\delta}{\lambda}$. By maximality, $Md^{n-1} \gtrsim |\Omega|$, and by the definition of Ω we can find δ -tubes T_j in the directions y_j such that $|T_j \cap E| > \lambda|T_j|$. Then

$$\int_E \sum_{j=1}^M \mathbb{1}_{T_j}(x) \, dx = \sum_{j=1}^M |T_j \cap E| \geq M\lambda\delta^{n-1}$$

and so there exists a point $x \in E$ belonging to at least $\frac{M\lambda\delta^{n-1}}{|E|}$ of the T_j s. By the Lemma, the sets $T_j \setminus \text{Ball}(x, C\delta/\varepsilon)$ for these j are disjoint. Hence

$$\begin{aligned} |E| &\geq \left| \bigcup_{T_j \ni x} E \cap T_j \setminus \text{Ball}(x, C\delta/\varepsilon) \right| \\ &= \sum_{T_j \ni x} |E \cap T_j \setminus \text{Ball}(x, \frac{1}{10}\lambda)| \\ &\geq \sum_{T_j \ni x} \frac{9}{10}\lambda|T_j| \\ &\geq \frac{M\lambda\delta^{n-1}}{|E|} \frac{9}{10}\lambda\delta^{n-1}. \end{aligned}$$

Rearranging this and putting in our lower bound for M gives $|\Omega| \lesssim \frac{|E|^2}{\delta^{n-1}\lambda^{n+1}}$ as required. \square

The theorem and its proof for \mathcal{N}_δ are exactly analogous. So the so-called “trivial bound” holds for *all* curves of the form (2.7), for both Keakeya and Nikodym

maximal functions. In particular, it is true for the “worst case” example of Bourgain. That example had $n = 3$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which gives curves of the form

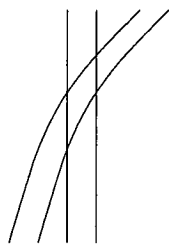
$$\begin{pmatrix} \omega_1 - 2ty_2 - 2t^2y_1 \\ \omega_2 - 2ty_1 \\ t \end{pmatrix}.$$

If we choose $\omega_1 = 0, \omega_2 = -2y_2$ then we see that each curve lies in the surface $x_1 = x_2x_3$. So the Kakeya set has dimension two, and for this A, B the trivial bound is in fact best possible.

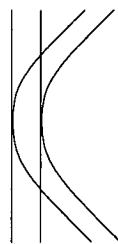
This suggests that $\frac{n+1}{2}$ for the set dimension and maximal function ought to correspond to the exponent $q = \frac{2(n+1)}{n-1}$ in Hörmander’s conjecture, since this is the result that is known to be true for all phases and cannot be improved for Bourgain’s example. However, the implication proved in Theorem 8 is weaker; one feels that the factor of two in the power of δ we obtained should not be there.

4.2.2 Failure for degenerate curves

The proof of the trivial bound also reveals the reason for the non-degeneracy criteria (2.2), (2.2’). Suppose that the latter does not hold, so that $\det(A + 2t_0B) = 0$ for some $t_0 \in [-1, 1]$. Then there exists $y \neq 0$ with $(A + 2t_0B)y = 0$. This means that the curves $\Gamma_0(0)$ (which is just the x_n -axis) and $\Gamma_y(0)$ meet at $x_n = 0$ and again at $x_n = 2t_0$. This might or might not lie in the range $x_n \in [-1, 1]$. Furthermore, the tangent to the curve $\Gamma_y(\omega)$ at height t is given



(a) Proper Intersection



(b) Tangential Intersection

Figure 4.1: *The intersection of two curved tubes*

by $(-2(A+2tB)y)$, and so at the height $x_n = t_0 \in [-1, 1]$ the tangent to $\Gamma_y(\omega)$ is vertical, as is that of $\Gamma_0(0)$. So without non-degeneracy, several curves in different directions and with different centre points ω may meet and share a tangent. This is bad, because the key point in the proof of the trivial bound was Lemma 20, which relied on the tubes intersecting “properly” as in Figure 4.1(a).

Let $U = \{y : (A + 2t_0B)y = 0\}$ and write $r = \dim U \geq 1$.

Theorem 22. *If the non-degeneracy criterion (2.2) fails, then \mathcal{K}_δ cannot be bounded from $L^{\frac{n+1}{2}}(\mathbb{R}^n)$ to $L^{n+1}(\mathbb{B}^{n-1})$ with any constant less than $\delta^{-\frac{n+2r-1}{n+r-1} \frac{n-1}{n+1}}$. In particular (4.1) fails for $p = \frac{n+1}{2}$.*

Proof: We expect, considering where the proof of the previous section breaks down, that we shall get a counterexample by making many tubes meet at the bad point. So pick a maximal δ -separated subset $\{y_j\}_{j=1}^M \subset \mathbb{B}^{n-1}$ and let $T_j = T_{y_j}^\delta(\omega_j)$ where $\omega_j = 2t_0Ay_j + 2t_0^2By_j$ so that all tubes pass through the point $(0, \dots, 0, t_0)$. Note also that $M \sim \delta^{-(n-1)}$.

Again we study how the tubes intersect. Given y_j , write $y_j = y_j^0 + y_j^\perp$ with $y_j^0 \in U$, $y_j^\perp \in U^\perp$. Then the distance of the curve Γ_j from the x_n -axis at height t is given by

$$\begin{aligned} |\omega_j - 2tAy_j - 2t^2By_j| &= |2t_0Ay_j + 2t_0^2By_j - 2tAy_j - 2t^2By_j| \\ &= 2|t_0 - t| |A(y_j^0 + y_j^\perp) + (t_0 + t)B(y_j^0 + y_j^\perp)| \\ &= 2|t_0 - t| |(A + (t_0 + t)B)y_j^\perp - (t_0 - t)By_j^0| \\ &\leq 2|t_0 - t| (\|A\| + \|B\|)|y_j^\perp| + 2|t_0 - t|^2 \|B\| |y_j^0|. \end{aligned}$$

Now this is at most δ if both $|t_0 - t| \lesssim \delta/|y_j^\perp|$ and $|t_0 - t| \lesssim \sqrt{\delta/|y_j^0|}$. To put this another way, T_j includes a δ -tube along the x_n -axis of length L provided that $|y_j^\perp| \lesssim \delta/L$ and $|y_j^0| \lesssim \delta/L^2$.

Choose $L = \delta^{\frac{n-1}{n+r-1}}$, which is between δ and 1. Then the δ -tube along the axis centre $(0, \dots, 0, t_0)$ and length L is included in at least

$$\frac{(\delta/L^2)^r (\delta/L)^{n-1-r}}{\delta^{n-1}} = \frac{1}{L^{r+n-1}} = \delta^{-(n-1)}$$

of the tubes (that is, virtually all of them). So we can then estimate

$$\begin{aligned} \left\| \sum_{j=1}^M \mathbb{1}_{T_j} \right\|_{\frac{n+1}{n-1}} &\gtrsim (\delta^{n-1} L (\delta^{-(n-1)})^{\frac{n+1}{n-1}})^{\frac{n-1}{n+1}} \\ &= \delta^{-\frac{n+2r-1}{n+r-1} \frac{n-1}{n+1}} (M \delta^{n-1})^{\frac{n}{n+1}} \end{aligned}$$

which, using Lemma 10, proves the theorem. \square

4.2.3 Minkowski result for any curves

Although the maximal function behaves differently for sets not satisfying the non-degeneracy criterion, the Hausdorff and Minkowski dimensions do not. This is intuitively clear, since a Kakeya set of degenerate curves includes a set of non-degenerate ones by simply removing a slice around the “bad” height. Shifting

and scaling part of what remains so that it lies in the region $x_n \in [-1, 1]$ gives a set of curves that falls within the scope of Theorem 21. So this subset, and hence the whole set, of the original curves has Minkowski and Hausdorff dimension at least $\frac{n+1}{2}$.

It is worth looking at a direct proof of this fact however, because it gives greater insight into the extra information encoded in the maximal function. To prove the trivial bound for the maximal function we needed Lemma 20 which gave a quantitative relation between the separation of tubes and the diameter of their intersection. For the Minkowski dimension version, however, we need only that the intersection of δ -separated tubes is at most a constant proportion of each tube—this is the idea that Tao has termed “shading” [23]. The weaker requirement means that we can deal with tangential intersections as in Figure 4.1(b) and with multiple intersections, so that we no longer need to assume the non-degeneracy criterion (2.2’).

Theorem 23. *Curved Kakeya sets in \mathbb{R}^n have Minkowski dimension at least $(n+1)/2$, even if (2.2’) is not satisfied.*

Proof: First we prove the following claim: There is a natural number k such that if the δ -tubes corresponding to $k\delta$ -separated y ’s intersect at height t_0 , then the tubes are disjoint over some fixed range of heights t .

By linearity in y and ω it is enough to show there exists k such that if $\Gamma_y(\omega)$ meets $\Gamma_0(0)$ (the x_n -axis) at height t_0 , then for some range of t independent of y and ω we have

$$|\omega - 2t(A + tB)y| > 2\delta.$$

But we know that $\omega = 2t_0(A + t_0B)$, so this becomes

$$|2(t_0 - t)(A + (t_0 + t)B)| > 2\delta$$

over some range of t , subject to $|y| > k\delta$. Let M be the matrix $(t_0 - t)(A + (t_0 + t)B)$. We must show $|My| > \frac{1}{k}|y|$. But this just amounts to finding a range of t (possibly depending on t_0) where the eigenvalues of M are bounded away from zero. This can certainly be done: Let $\eta_t > 0$ be the minimum absolute value of the eigenvalues of $A + (t_0 + t)B$, where $t \in (-1, 1)$ is not a solution of $\det(A + (t_0 + t)B) = 0$. The set of such solutions is finite. If the set $\{t : \eta_t > \frac{1}{n}\}$ had measure zero for all n , then the interval $(-1, 1)$ would have zero measure also. So there is some set of t of positive measure where the eigenvalues are all bounded away from zero. This proves the claim.

Now let E be a curved Kakeya set and let $\text{nbd}_\delta(E)$ denote its δ -neighbourhood. By the engulfing property (Lemma 7) we may assume that this consists of

at most $C\delta^{-(n-1)}$ tubes whose directions are $k\delta$ -separated. Write $M = \|\sum \mathbb{1}_T\|_\infty$ and let $x_0 = (x'_0, t_0) \in \mathbb{R}^n$ be a point where this is attained. By the claim, the tubes passing through this point are mutually disjoint over a non-null set of heights. Hence

$$|\text{nb}\delta(E)| \geq \left| \bigcup_{T \ni x_0} T \right| \geq cM\delta^{n-1}.$$

But also

$$1 = \sum |T| = \int \sum \mathbb{1}_T \leq \left\| \sum \mathbb{1}_T \right\|_\infty |\cup T| = M |\text{nb}\delta(E)|.$$

So

$$|\text{nb}\delta(E)| \geq \max \left\{ \frac{1}{M}, cM\delta^{d-1} \right\} \geq C\delta^{\frac{n-1}{2}}$$

so that $\dim(E) \geq (n+1)/2$ as required. \square

This result is rather curious, since it gives us an example where the maximal version of the conjecture is genuinely stronger than the Minkowski dimension. It shows that a Minkowski bound merely needs that different curves do not meet too often, whereas a maximal function bound requires that whenever they do meet, they have different tangents.

Chapter 5

Geometric Methods

*Oh, this is just a sheep in Wolff's clothing, or a Wolff in sheep's clothing, or mutton dressed
as lamb or something like that!*
Tony Carbery

5.1 Introduction

In this chapter we consider the use of geometric techniques in proving non-trivial results about curves. Such techniques have a long history in the straight line case—indeed the earliest non-trivial result, namely the optimal bound for the Nikodym maximal function in two dimensions, was obtained by Córdoba using the geometric fact that two lines meet in a point. Later results of Bourgain and Wolff, and still more recently those of Katz, Laba and Tao, built on this work by grouping the lines into two-planes and using Córdoba's result plus more sophisticated techniques.

With curves the situation is much more complicated. Although the two dimensional case, as we have seen, is no harder for curves, when we move into higher dimensions many things can happen. Curves cannot be grouped into planes in any natural way—the obvious analogue uses surfaces, but then there are many surfaces that pass through two given curves. As we shall see, the proofs by different authors for the straight line case each suggest a different definition of the surfaces. Matters are complicated further by the many ways that two surfaces can intersect: While a pair of planes in different directions must intersect properly in a line, two different surfaces might miss each other, or share a tangent plane at a point, or meet along a curve which might well not be one of the family of curves under consideration.

With these things in mind it is hardly surprising that some of our attempts to modify the straight-line proofs will fail. However, we do have one small success: a result for the Nikodym maximal function implying the lower bound of $\frac{n+2}{2}$ for

Nikodym sets of a special class of parabolas. This result, which uses ideas due to Katz, will be proved first. We shall then go on to discuss the approaches of Bourgain and Wolff, explaining the possibilities and barriers in the curved case.

5.2 Katz's approach

Some time after Wolff's paper giving the $\frac{n+2}{2}$ bound, Katz [19] gave a new proof which seems more elementary, in that it isolates the geometry showing that the main fact is that a triangle lies in a plane, and the remainder of the argument is a simple (but clever) splitting up of the linearised maximal function into bounded pieces.

This argument has been said to be no more than a reworking of Wolff's paper, but as we shall see, the situation with curves makes the two look very different.

Our result is the following:

Theorem 24. *The Nikodym maximal function \mathcal{N}_δ satisfies the bound*

$$\|\mathcal{N}_\delta\|_{n \rightarrow n} \leq C(\log 1/\delta)^\alpha \delta^{-\frac{n-2}{2n}}$$

where C and α are some positive numbers, provided that the curves under consideration are parabolas with $BA^{-1}B = 0$.

By Theorem 11 this implies that the Nikodym sets of these curves have Hausdorff and Minkowski dimension at least $\frac{n+2}{2}$.

The condition on the matrices A and B arises naturally in the proof as we shall see in Lemma 26. This class of curves seems to be particularly amenable to the proof methods that have been used in the straight line case, since further results for these curves will be obtained by the arithmetic methods in Chapter 6.

We shall actually prove Theorem 24 for the linearised version of the Nikodym maximal function $\mathcal{L}\mathcal{N}_\delta$ —recall its definition from page 26: We have divided \mathbb{R}^{n-1} into δ -cubes Q_j where j runs over $\mathbb{B}^{n-1} \cap \delta\mathbb{Z}^{n-1}$. To each index j we have an associated curved tube $T_j = T_{y_j}(\omega_j)$ where $\omega_j \in Q_j$ and y_j is arbitrary. Then

$$\mathcal{L}\mathcal{N}_\delta f(\omega) = \sum_j \mathbb{1}_{Q_j}(\omega) \frac{1}{|T_j|} \int_{T_j} f(x) \, dx.$$

Of course, we must seek bounds that are independent of the choice of the tubes. We shall also need to define related functions where the index set is specified:

$$\mathcal{L}\mathcal{N}_\mathbb{A} f(\omega) = \sum_{j \in \mathbb{A}} \mathbb{1}_{Q_j}(\omega) \frac{1}{|T_j|} \int_{T_j} f(x) \, dx.$$

As in Wolff's approach, the main geometric object considered is the *hairbrush*:

Definition. Let \mathbb{A} be a finite set of indices $j \in \delta\mathbb{Z}^{n-1}$. A *hairbrush* is a set $\mathbb{H} \subseteq \mathbb{A}$ such that there exists some curved $1 \times \delta$ tube T that intersects all T_i with $i \in \mathbb{H}$.

Note that the central tube T can be any curved tube of the family, not necessarily one of those associated to some j .

Much of the geometry of the situation is encoded in the behaviour of these hairbrushes, in the form of the following lemma:

Lemma 25 (Hairbrush Lemma). *If the curves are parabolas with $BA^{-1}B = 0$, then for all hairbrushes \mathbb{H} we have $\|\mathcal{LN}_{\mathbb{H}}\|_{n \rightarrow n} \leq C(\log 1/\delta)^\alpha$.*

The proof of this will involve surfaces, and will show why we are able to handle only a restricted class of curves. But given the lemma, we can prove the theorem just as in the straight line case, by splitting up the operator into many sums.

Proof of Theorem 24: It is enough to prove a weak type estimate for \mathcal{LN} , since this implies strong type at the cost of an additional \log [8, p. 48]. By the covering lemma (Lemma 10) the theorem is true if and only if

$$\begin{aligned} & \left\| \sum_{j \in \mathbb{A}} \mathbb{1}_{T_j} \right\|_{n'} \lesssim (\log 1/\delta)^\alpha \delta^{-\frac{n-2}{2n}} (\delta^{n-1} \# \mathbb{A})^{1/n'} \\ \iff & \int_{\mathbb{R}^n} \left(\sum_{j \in \mathbb{A}} \mathbb{1}_{T_j}(x) \right)^{\frac{n}{n-1}} dx \lesssim (\log 1/\delta)^\alpha \delta^{-\frac{n-2}{2(n-1)}} (\delta^{n-1} \# \mathbb{A}) \\ \iff & \sum_{j \in \mathbb{A}} \frac{1}{|T_j|} \int_{T_j} \left(\sum_{i \in \mathbb{A}} \mathbb{1}_{T_i}(x) \right)^{\frac{1}{n-1}} dx \lesssim (\log 1/\delta)^\alpha \delta^{-\frac{n-2}{2(n-1)}} \# \mathbb{A} \end{aligned}$$

Denote the quantity appearing in the first sum by $M_{\mathbb{A}}$, that is

$$M_{\mathbb{A}}(j) := \frac{1}{|T_j|} \int_{T_j} \left(\sum_{i \in \mathbb{A}} \mathbb{1}_{T_i}(x) \right)^{\frac{1}{n-1}} dx.$$

We would like to subdivide this quantity into dyadic scales, by considering those i that are at distance between 2^{-k} and $2^{-(k+1)}$ from j . Note that by elementary properties of sequences of positive reals, the sum over k can then be pulled out of the integral. But what remains then depends only on pairs i, j with $|i - j| \sim 2^{-k}$. Let P be a cube in \mathbb{R}^{n-1} of side 10×2^{-k} . Note that this cube is larger than the cubes Q_j since $\delta < 2^{-k}$. It then suffices, for every choice of P , to obtain the estimate

$$\begin{aligned} \sum_{j \in \mathbb{A} \cap P} M_k(j) &:= \sum_{j \in \mathbb{A} \cap P} \frac{1}{|T_j|} \int_{T_j} \left(\sum_{\substack{i \in \mathbb{A} \cap P \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) \right)^{\frac{1}{n-1}} dx \\ &\lesssim (\log 1/\delta)^\alpha \delta^{-\frac{n-2}{2(n-1)}} \#(\mathbb{A} \cap P) \end{aligned}$$

and then sum over P and k . Both sums have only logarithmically many terms.

The next stage is to find as many large hairbrushes in $\mathbb{A} \cap P$ as possible, where *large* means of cardinality at least N , to be chosen later. So, if there exists some curved tube T (of the form $T_y(\omega)$ but not necessarily one of the T_j) such that there are at least N elements $i \in \mathbb{A} \cap P$ with $T \cap T_i \neq \emptyset$, then call these elements \mathbb{H}_1 . Then look for another large hairbrush in the remaining elements $\mathbb{A} \cap P \setminus \mathbb{H}_1$. Eventually there are no more hairbrushes, so call the remaining bad elements \mathbb{B} . This constructs hairbrushes $\mathbb{H}_1, \dots, \mathbb{H}_m$ each of cardinality at least N , and a bad set $\mathbb{B} := \mathbb{A} \setminus (\mathbb{H}_1 \cup \dots \cup \mathbb{H}_m)$. Let $\mathbb{H} := \mathbb{H}_1 \cup \dots \cup \mathbb{H}_m$.

Since the hairbrushes are disjoint sets of indices (although the tubes they correspond to may well not be), and $\mathbb{A} \cap P$ has at most $2^{-k(n-1)}\delta^{-(n-1)}$ elements, it follows that $m \leq 2^{-k(n-1)}\delta^{-(n-1)}/N$.

Now split the sum into four pieces

$$\sum_{j \in \mathbb{A} \cap P} M_k(j) \leq \sum_{j \in \mathbb{H}} M_{k, \mathbb{H}}(j) + \sum_{j \in \mathbb{B}} M_{k, \mathbb{H}}(j) + \sum_{j \in \mathbb{H}} M_{k, \mathbb{B}}(j) + \sum_{j \in \mathbb{B}} M_{k, \mathbb{B}}(j)$$

where

$$M_{k, \mathbb{H}}(j) := \frac{1}{|T_j|} \int_{T_j} \left(\sum_{\substack{i \in \mathbb{H} \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) \right)^{\frac{1}{n-1}} dx$$

$$M_{k, \mathbb{B}}(j) := \frac{1}{|T_j|} \int_{T_j} \left(\sum_{\substack{i \in \mathbb{B} \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) \right)^{\frac{1}{n-1}} dx.$$

The first sum is estimated using the hairbrush lemma. For

$$\begin{aligned} \|\mathcal{LN}_{\mathbb{H}} f\|_n^n &= \int \left(\sum_i \mathcal{LN}_{\mathbb{H}_i} f(\omega) \right)^n d\omega \\ &= \sum_i \int (\mathcal{LN}_{\mathbb{H}_i} f(\omega))^n d\omega \quad \text{since each } \omega \text{ gives only one non-zero term} \\ &\leq m \|\mathcal{LN}_{\mathbb{H}_i}\|_{n \rightarrow n}^n \|f\|_n^n \end{aligned}$$

showing that $\|\mathcal{LN}_{\mathbb{H}}\|_{n \rightarrow n} \leq C m^{1/n} (\log 1/\delta)^\alpha$ by Lemma 25. Then by the covering lemma we obtain

$$\sum_{j \in \mathbb{H}} M_{k, \mathbb{H}}(j) \leq (C m^{1/n} (\log 1/\delta)^\alpha)^{n'} \# \mathbb{H}.$$

For the second sum

$$\begin{aligned}
\sum_{j \in \mathbb{B}} M_{k, \mathbb{H}}(j) &= \sum_{j \in \mathbb{B}} \frac{1}{|T_j|} \int_{T_j} \left(\sum_{\substack{i \in \mathbb{H} \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) \right)^{\frac{1}{n-1}} dx \\
&\leq \sum_{j \in \mathbb{B}} \left(\frac{1}{|T_j|} \int_{T_j} \sum_{\substack{i \in \mathbb{H} \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) dx \right)^{\frac{1}{n-1}} && \text{by Jensen} \\
&\leq (\#\mathbb{B})^{1-\frac{1}{n-1}} \left(\sum_{j \in \mathbb{B}} \frac{1}{|T_j|} \int_{T_j} \sum_{\substack{i \in \mathbb{H} \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) dx \right)^{\frac{1}{n-1}} && \text{by Hölder} \\
&\leq (\#\mathbb{B})^{1-\frac{1}{n-1}} \left(\sum_{i \in \mathbb{H}} \sum_{\substack{j \in \mathbb{B} \\ |i-j| \sim 2^{-k}}} \frac{1}{|T_j|} |T_i \cap T_j| \right)^{\frac{1}{n-1}} && \text{by swapping sums} \\
&\lesssim (\#\mathbb{B})^{1-\frac{1}{n-1}} \left(\sum_{i \in \mathbb{H}} \sum_{\substack{j \in \mathbb{B} \\ T_i \cap T_j \neq \emptyset}} \frac{1}{\delta^{n-1}} \delta^n 2^k \right)^{\frac{1}{n-1}} && \text{by Lemma 20} \\
&\leq (\#\mathbb{B})^{1-\frac{1}{n-1}} \left(\sum_{i \in \mathbb{H}} N \delta 2^k \right)^{\frac{1}{n-1}} && \text{since no large hairbrushes in } \mathbb{B} \\
&= (\#\mathbb{B})^{1-\frac{1}{n-1}} (\#\mathbb{H} N \delta 2^k)^{\frac{1}{n-1}} \\
&\leq (\#\mathbb{A} \cap P) (N \delta 2^k)^{\frac{1}{n-1}}.
\end{aligned}$$

The third and fourth sums can be tackled together, since for *all* $j \in \mathbb{A} \cap P$ we have

$$\begin{aligned}
M_{k, \mathbb{B}}(j) &:= \frac{1}{|T_j|} \int_{T_j} \left(\sum_{\substack{i \in \mathbb{B} \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) \right)^{\frac{1}{n-1}} dx \\
&\leq \left(\frac{1}{|T_j|} \int_{T_j} \sum_{\substack{i \in \mathbb{B} \\ |i-j| \sim 2^{-k}}} \mathbb{1}_{T_i}(x) dx \right)^{\frac{1}{n-1}} && \text{by Jensen} \\
&= \left(\frac{1}{|T_j|} \sum_{\substack{i \in \mathbb{B} \\ |i-j| \sim 2^{-k}}} |T_i \cap T_j| \right)^{\frac{1}{n-1}} \\
&\lesssim \left(\frac{1}{\delta^{n-1}} \sum_{\substack{i \in \mathbb{B} \\ T_i \cap T_j \neq \emptyset}} \delta^n 2^k \right)^{\frac{1}{n-1}} && \text{by Lemma 20} \\
&\leq (N \delta 2^k)^{\frac{1}{n-1}} && \text{since no large hairbrushes in } \mathbb{B}.
\end{aligned}$$

So the last three sums all give an estimate of $\#(\mathbb{A} \cap P) (N \delta 2^k)^{\frac{1}{n-1}}$, while the first, after putting in the upper bound for m , gives $\#\mathbb{H} (2^{-k(n-1)} \delta^{-(n-1)} / N)^{\frac{1}{n-1}}$.

We optimally choose $N = 2^{-k\frac{n}{2}}\delta^{-\frac{n}{2}}$ and add the four pieces to obtain

$$\sum_{j \in \mathbb{A} \cap P} M_k(j) \lesssim 2^{-k\frac{n-2}{2(n-1)}} (\log 1/\delta)^\alpha \delta^{-\frac{n-2}{2(n-1)}} \#(\mathbb{A} \cap P)$$

which gives the result after summing over all P of side 2^{-k} and all k . \square

This proof required no geometry other than the estimate for the diameter of the intersection of two tubes (Lemma 20) and that contained in the hairbrush lemma. In the straight line case, the proof of the latter reduces to the fact that a triangle lies in a plane, and moreover knowing two of the sides is enough to specify which plane it is. So before we go on to prove Lemma 25 we must study surfaces that are determined by two curves forming the sides of a “triangle”.

By the linearity of (2.7) in y and ω , we may assume that one of the given curves is $\Gamma_0(0)$. Let the other be $\Gamma_{y_0}(\omega_0)$ and assume that they meet at height t_0 . The surface is the locus of those curves $\Gamma_y(\omega)$ that meet the first at s and the second at u . Note that none of these three heights are equal, since the curves are never horizontal, and we must exclude the possibility of $\Gamma_y(\omega)$ meeting the two given curves at their common point, since this would allow every curve to belong to the locus. This is made clearer by the following picture:

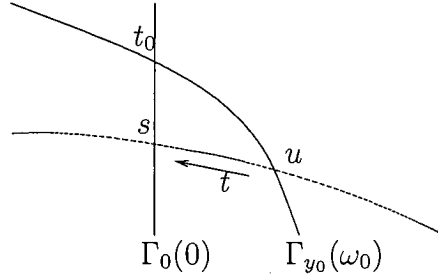


Figure 5.1: Notation for a curved triangle

Now we have the following equations:

$$0 = \omega_0 - 2t_0Ay_0 - 2t_0^2By_0 \quad (5.1)$$

$$0 = \omega - 2sAy - 2s^2By \quad (5.2)$$

$$\omega_0 - 2uAy_0 - 2u^2By_0 = \omega - 2uAy - 2u^2By \quad (5.3)$$

Subtracting (5.2) from (5.3) we find that

$$y = \frac{1}{2(s-u)} (A + (s+u)B)^{-1} (\omega_0 - 2uAy_0 - 2u^2By_0)$$

which is well defined because $s \neq u$ and by (2.2'). Substitute into (5.2) to find ω :

$$\omega = \frac{s}{s-u} (A + sB) (A + (s+u)B)^{-1} (\omega_0 - 2uAy_0 - 2u^2By_0).$$

(5.1) has not been used yet, so we use it to eliminate ω_0 :

$$\omega = \frac{2s(t_0 - u)}{s - u} (A + sB) (A + (s + u)B)^{-1} (A + (t_0 + u)B) y_0.$$

Finally substitute this y and ω into (2.7) to obtain

$$\left(\frac{2(s-t)(t_0-u)}{s-u} (A + (s+t)B) (A + (s+u)B)^{-1} (A + (t_0+u)B) y_0 \right) =: \Sigma(s, t, u) \quad (5.4)$$

as the parametrisation of the locus we are interested in. Note that if $B = 0$ then this reduces to the plane $\begin{pmatrix} r^{Ay_0} \\ t \end{pmatrix}$ as expected.

However, in general we have three parameters (s, u, t) and so the locus is a three-dimensional object, not a surface at all. The following lemma determines when this is so.

Lemma 26. *The locus described by (5.4) is a surface if and only if either B is a multiple of A , or $BA^{-1}B = 0$.*

Proof: We get a surface if and only if at every point the three tangent vectors $\frac{\partial \Sigma}{\partial s}, \frac{\partial \Sigma}{\partial t}, \frac{\partial \Sigma}{\partial u}$ are coplanar. Because of the t in the last component of Σ , this happens if and only if the two vectors consisting of the first $n - 1$ components of $\frac{\partial \Sigma}{\partial s}$ and $\frac{\partial \Sigma}{\partial u}$ are parallel. These are

$$\begin{aligned} \frac{2(t_0 - u)}{s - u} \left[(s - t)(B - (A + (s + t)B)(A + (s + u)B)^{-1}B) \right. \\ \left. + \frac{t - u}{s - u} (A + (s + t)B) \right] (A + (s + u)B)^{-1} (A + (t_0 + u)B) y_0 \end{aligned}$$

and

$$\begin{aligned} \frac{2(s - t)}{s - u} (A + (s + t)B) (A + (s + u)B)^{-1} \left[\frac{t_0 - s}{s - u} (A + (t_0 + u)B) + \right. \\ \left. (t_0 - u)(B - B(A + (s + u)B)^{-1}(A + (t_0 + u)B)) \right] y_0. \end{aligned}$$

For all possible loci to be surfaces, we need this for all y_0 , so that in fact the matrices themselves must be “parallel”, by which we mean that one is a scalar multiple of the other. Next we may rewrite the above, but ignore the initial (scalar) function of (s, t, u) and multiply on the left by $(A + (s + u)B)(A + (s + t)B)^{-1}$ and on the right by $(A + (t_0 + u)B)^{-1}(A + (s + u)B)$. We thus require the following two expressions to be parallel:

$$\begin{aligned} \frac{t - u}{s - u} (A + (s + u)B) + (s - t)(u - t)B(A + (s + t)B)^{-1}B \\ \frac{t_0 - s}{s - u} (A + (s + u)B) + (t_0 - u)(s - t_0)B(A + (t_0 + u)B)^{-1}B \end{aligned}$$

This is true for all s, t, t_0, u provided that none of these coincide. So it is allowable to substitute $t = 0$ and $u = s - t_0$, which leads to scalar multiples of

$$(A + (2s - t_0)B) - st_0B(A + sB)^{-1}B \quad (5.5)$$

$$(A + (2s - t_0)B) - t_0(2t_0 - s)B(A + sB)^{-1}B. \quad (5.6)$$

Any linear combinations of these two expressions must also be parallel. In particular

$$\begin{aligned} (5.5) - (5.6) &= 2t_0(t_0 - s)B(A + sB)^{-1}B \\ (2t_0 - s)(5.5) - s(5.6) &= 2(t_0 - s)(A + (2s - t_0)B) \end{aligned}$$

Now the second of these is an invertible matrix. If B is also invertible, by rearranging we find that $B \parallel A$. If not, then we must have $B(A + sB)^{-1}B = 0$ for all s , and hence $BA^{-1}B = 0$

These are necessary conditions. Clearly B a multiple of A is sufficient since (5.4) is just the plane $\begin{pmatrix} rA y_0 \\ t \end{pmatrix}$. In the other case, we know that $(A + zB)^{-1} = A^{-1} - zA^{-1}BA^{-1}$ which makes all the matrices cancel down to give

$$\begin{pmatrix} r(A + (t_0 + t)B)y_0 \\ t \end{pmatrix} \quad (5.7)$$

which is indeed a surface. \square

We are now ready to prove the Hairbrush Lemma, and hence complete the proof of Theorem 24.

Proof of Lemma 25: We have a set \mathbb{H} of indices which forms a hairbrush with central tube T . By linearity assume that the central tube is $T_0(0)$. Denote the other tubes by $T_j = T_{y_j}(\omega_j)$, where $\omega_j \in Q_j$ and so $\omega_j \approx j$. We partition the set \mathbb{H} in several ways. First, let H_k be the set of all those indices whose tubes meet T at “angle” 2^{-k} ; that is,

$$H_k := \{i \in \mathbb{H} : |y_i| \sim 2^{-k}\}.$$

For a fixed ω , there can be only one k such that $\mathcal{LN}_{H_k}(\omega) \neq 0$, so it is enough to prove $\|\mathcal{LN}_{H_k}\| \leq C(\log 1/\delta)^\alpha$ since there are only logarithmically many k . Then by the arguments used previously, this bound is true if and only if

$$\sum_{i \in H_k} M_{H_k}(i) \leq C(\log 1/\delta)^\alpha \#H_k.$$

For fixed $j \in H_k$ split up H_k into further sets $H_{j,k,l,m}$ as follows:

$$H_{j,k,l,m} := \{i \in H_k : |y_i - y_j| \sim 2^{-l} \text{ and } \text{dist}(T_i \cap T_j, T_j \cap T) \sim \delta 2^{l+m}\}.$$

Note that this set is empty unless $l \geq k - 2$. Now it is enough to show that $M_{H_j, k, l, m}(j) \leq C(\log 1/\delta)^\alpha$, because

$$M_{H_k}(i) \leq \sum_{j \in H_k} \sum_l \sum_m M_{H_j, k, l, m}(i)$$

and there are only logarithmically many l and m and the sum over j introduces a factor $\#H_k$.

Next comes the geometric part of the argument. We need to show that $\#\mathbb{H}_{j, k, l, m}$ is not too big, which means that given the central tube T and another fixed tube $T_j : j \in H_k$ there are few other tubes T_i meeting these with all the correct “angles” and distances. In the straight line case this follows from simple consideration of similar triangles as in Figure 5.2.

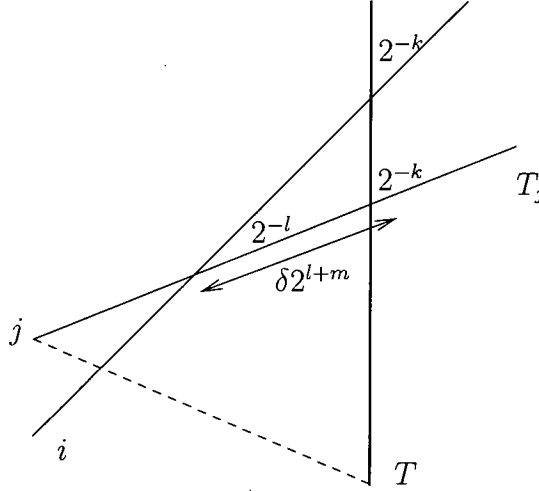


Figure 5.2: In the straight line case, by similar triangles we have $|i-j| \lesssim 2^{-l}$ and $\text{dist}(i, \text{line}) \lesssim 2^{-(l+m)}$.

In the curved case, the dotted line in the picture is instead the curve of intersection of the base plane $x_n = 0$ with the surface determined by T and T_j . Since we cannot appeal to similar triangles with curves, we state and prove our claim more formally:

Claim. Let A and B satisfy $BA^{-1}B = 0$. Suppose that we are given three curved tubes $T = T_0(0)$, $T_j = T_{y_j}(\omega_j)$ and $T_i = T_{y_i}(\omega_i)$ with $|y_j|, |y_i| \in (\frac{1}{2^{k+1}}, \frac{1}{2^k})$ and $|y_j - y_i| \in (\frac{1}{2^{l+1}}, \frac{1}{2^l})$. Here $l \geq k - 2$ and all the powers of 2 that occur are greater than δ . Suppose that T_j meets the axis at height t_j , T_i meets it at t_i , and they meet each other at s , where $\delta 2^{l+m} \leq |s - t_j| \leq \delta 2^{l+m+1}$. Then $|\omega_j - \omega_i| \leq \frac{1}{2^l}$, and ω_i is at distance at most $\frac{1}{2^{l+m}}$ from the intersection of the surface (5.7) with the horizontal plane.

We have the following equations

$$\begin{aligned}\omega_j &= 2t_j(A + t_j B)y_j \\ \omega_i &= 2t_i(A + t_i B)y_i + \varepsilon \\ \omega_i - 2s(A + sB)y_i &= \omega_j - 2s(A + sB)y_j + \eta\end{aligned}$$

where ε and η are errors due to the thickness of the tubes, and are of order at most δ . The first assertion is easy:

$$\begin{aligned}|\omega_j - \omega_i| &= |2s(A + sB)(y_j - y_i) - \eta| \\ &\leq C|y_j - y_i| + |\eta| \\ &\lesssim \frac{1}{2^l}\end{aligned}$$

For the second, begin by eliminating the ω s:

$$2(t_j - s)(A + (t_j + s)B)y_j + \eta = 2(t_i - s)(A + (t_i + s)B)y_i + \varepsilon \quad (5.8)$$

This can be rearranged to give y_i in terms of y_j . Then plug this back in to the 2nd of our original equations to give

$$\begin{aligned}\omega_i &= \frac{2t_i(t_j - s)}{t_i - s}(A + t_i B)(A + (t_i + s)B)^{-1}(A + (t_j + s)B)y_j \\ &\quad + \frac{t_i}{t_i - s}(A + t_i B)(A + (t_i + s)B)^{-1}(\eta - \varepsilon) + \varepsilon \\ &= \frac{2t_i(t_j - s)}{t_i - s}(A + t_j B)y_j + \frac{t_i}{t_i - s}(I - sBA^{-1})(\eta - \varepsilon) + \varepsilon\end{aligned}$$

where we have used the fact that $BA^{-1}B = 0$. Looking back at (5.7) we discover that the first term belongs to the intersection of the surface determined by the first two curves with the horizontal plane. So the distance we are interested in is at most the absolute value of the other two terms, so at most $\frac{C}{|t_i - s|}\delta + \delta$. Finally we just ensure that $|t_i - s|$ is comparable to $|t_j - s|$. From (5.8) using the usual argument about eigenvalues being bounded above and below, we get

$$\begin{aligned}C|t_i - s|2^{-k} &\geq c\delta 2^{l+m}2^{-k} - \delta \\ |t_i - s| &\geq c\delta 2^{l+m} - \delta 2^k \\ &\gtrsim \delta 2^{l+m}\end{aligned}$$

provided that $k - l + m$ is not too large. Since $l \geq k - 2$ this could happen only with l close to k and m small, in which case the claim is trivial anyway. So the distance of ω_i from the curve of intersection is at most $2^{-(l+m)}$ and we have proved the claim.

We can now complete the proof of the Hairbrush Lemma, and hence the whole Theorem. The claim tells us that

$$\#H_{j,k,l,m} \lesssim 2^{-l} (2^{-(l+m)})^{n-2} \delta^{n-1}.$$

which we use as follows:

$$\begin{aligned}
M_{H_{j,k,l,m}}(i) &:= \frac{1}{|T_i|} \int_{T_i} \left(\sum_{p \in H_{j,k,l,m}} \mathbb{1}_{T_p}(x) \right)^{\frac{1}{n-1}} dx \\
&= \frac{1}{|T_i|} \int_{\{x \in T_i : \text{dist}(x, T_i \cap T) \sim \delta 2^{l+m}\}} \left(\sum_{p \in H_{j,k,l,m}} \mathbb{1}_{T_p}(x) \right)^{\frac{1}{n-1}} dx \quad \text{by defn. of } H_{j,k,l,m} \\
&\leq \frac{1}{|T_i|} \left| \{x \in T_i : \text{dist}(x, T_i \cap T) \sim \delta 2^{l+m}\} \right|^{1-\frac{1}{n-1}} \\
&\quad \times \left(\int_{T_i} \sum_{p \in H_{j,k,l,m}} \mathbb{1}_{T_p}(x) \right)^{\frac{1}{n-1}} dx \quad \text{by Hölder} \\
&\leq \frac{1}{\delta^{n-1}} (\delta^n 2^{l+m})^{1-\frac{1}{n-1}} \left(\sum_{p \in H_{j,k,l,m}} |T_i \cap T_p| \right)^{\frac{1}{n-1}} \\
&\lesssim \frac{1}{\delta^{n-1}} (\delta^n 2^{l+m})^{1-\frac{1}{n-1}} (\#H_{j,k,l,m} \delta^n 2^l)^{\frac{1}{n-1}} \quad \text{by Lemma 20} \\
&\lesssim \frac{1}{\delta^{n-1}} (\delta^n 2^{l+m})^{1-\frac{1}{n-1}} \left(2^{-l} (2^{-(l+m)})^{n-2} \delta^{n-1} \delta^n 2^l \right)^{\frac{1}{n-1}} \quad \text{by the claim} \\
&= 1.
\end{aligned}$$

Summing over all the index sets gives the result. \square

The proof did not really require the whole locus of curves meeting two given ones to be a surface, but merely that the intersection of the locus with the horizontal plane is a one-parameter curve. However, close examination of the proof of Lemma 26 shows that this still requires $BA^{-1}B = 0$.

In Chapter 7 we shall look more closely at the condition $BA^{-1}B = 0$, and show that it is linked, surprisingly, with Bourgain's "Worst Case" theorem (Theorem 5).

So we now have a non-trivial result for the Nikodym problem with parabolas, or equivalently for Kakeya with hyperbolas. It is natural to ask whether a similar result for the other two problems (recall the diagram of Figure 2.2) might hold. The answer is no. The difficulty lies in the proof of the claim on page 63. There we would have the same equations, but in our hypotheses and conclusions we would have to swap the roles of y and ω . Instead of showing that ω_i is close to the surface, we would have to show that y_i is close to a direction of a curve lying in the surface. This is undoubtedly true, since after all, the tube T_i by definition is one that meets both T and T_j . However, the key point is not so much whether

the locus is a genuine surface rather than three-dimensional, but rather that the set of ω_i , or in the Takeya case y_i , in the surface is a one-parameter family. The following result shows why the Nikodym and Takeya cases are clearly different.

Proposition 27. *Suppose that the curve $\Gamma_y(\omega)$ is included in the locus (5.4). Then*

- (i) *The point ω must belong to a family described by only one parameter if and only if $B \parallel A$ or $BA^{-1}B = 0$, in which cases $\omega = r(A + t_0B)y_0$ for some r .*
- (ii) *The direction y must belong to a family described by only one parameter if and only if we have the trivial case $B \parallel A$.*

Proof: From

$$\omega - 2t(A + tB)y = \frac{2(s - t)(t_0 - u)}{s - u} (A + (s + t)B) (A + (s + u)B)^{-1} (A + (t_0 + u)B)y_0$$

we obtain

$$\begin{aligned} \omega &= \frac{2s(t_0 - u)}{s - u} (A + sB) (A + (s + u)B)^{-1} (A + (t_0 + u)B)y_0 \\ y &= \frac{2(t_0 - u)}{s - u} (A + (s + u)B)^{-1} (A + (t_0 + u)B)y_0 \end{aligned}$$

where for a fixed curve s and u will depend on t . However, we are considering the sets of all such y and ω , so we allow s and u to vary. We also require the property for all y_0 and t_0 .

- (i) To show that the locus of all ω is a curve we require that the derivatives with respect to s and u are always parallel. We have calculated these already in the proof of Lemma 26, and the conclusion there leads to that claimed above.

- (ii) For y , the two derivatives are

$$\begin{aligned} &\frac{1}{s - u} (A + (s + u)B)^{-1} \left[\frac{t_0 - s}{s - u} (A + (t_0 + u)B) \right. \\ &\quad \left. + (t_0 - u)B(I - (A + (s + u)B)^{-1}(A + (t_0 + u)B)) \right] \\ &\frac{t_0 - s}{s - u} (A + (s + u)B)^{-1} \left[-B(A + (s + u)B)^{-1} \right. \\ &\quad \left. - \frac{1}{s - u} I \right] (A + (t_0 + u)B). \end{aligned}$$

Ignoring scalar functions and multiplying by invertible matrices on the right and left, we thus require the following two expressions to be parallel:

$$\begin{aligned} &A + (s + u)B - (t_0 - u)(s - u)B(A + (t_0 + u)B)^{-1}B \\ &A + 2uB \end{aligned}$$

Setting $u = -t_0$ and subtracting gives

$$(s + t_0)BA^{-1}[A - 2t_0B] \parallel A - 2t_0B$$

and since $A - 2t_0B$ is invertible by (2.2'), $BA^{-1} \parallel I$. Hence B is a (possibly zero) multiple of A .

In order to convince ourselves, we check that if $BA^{-1}B = 0$, then y is given by $\frac{2(t_0-u)}{s-u}(I + (t_0 - s)A^{-1}B)y_0$, which does have two parameters unless $B \parallel A$. \square

So theakeya and Nikodym cases have an important difference when approached in this way. In particular, there can be no analogue of the claim on page 63 for theakeya version with parabolas.

Apart from certain steps in the proof not working, we already know that the result is false in theakeya parabola version, for the example of Theorem 5 satisfies $BA^{-1}B = 0$ and we saw on page 51 that that the lower bound of $\frac{n+2}{2}$ for theakeya set dimension does not hold.

5.3 Bourgain's approach

The earliest non-trivial bound for theakeya maximal function in dimension greater than two was the optimal bound from $L^{7/3}(\mathbb{R}^3)$, which was proved in 1991 by Bourgain [2]. His argument used the idea of "bushes" of tubes (which we met earlier in the proof of the trivial bound) together with Córdoba's L^2 result for theakeya maximal function in the plane. He was also able to obtain non-trivial results in higher dimensions by the same method, by inductively applying the result for one lower dimension each time.

In this chapter we look at what happens when we try to apply similar techniques to curves. We have not yet been able to obtain the bound, but have reduced the problem to a possibly simpler one, namely an L^2 bound for a maximal function related to curved surfaces. We define this now.

Suppose that we have a fixed curve $\Gamma_{\bar{y}}(\bar{\omega})$ and a point a lying away from it. Define a surface $\Sigma(\bar{y}, \bar{\omega}, a)$ as the locus of all curves through a that intersect $\Gamma_{\bar{y}}(\bar{\omega})$. Denoting one of these curves by $\Gamma_y(\omega)$ as usual, and letting s denote the height where the two curves meet, we have the following equations:

$$\begin{aligned} a' &= \omega - 2a_n A y - 2a_n^2 B y \\ \bar{\omega} - 2s A \bar{y} - 2s^2 B \bar{y} &= \omega - 2s A y - 2s^2 B y \end{aligned}$$

Solving these we find that

$$\begin{aligned}\omega &= a' + 2a_n(A + 2a_nB)y \\ y &= \frac{1}{2(a_n - s)}(A + (a_n + s)B)^{-1}[\bar{\omega} - a' - 2s(A + sB)\bar{y}]\end{aligned}$$

and hence the surface $\Sigma(\bar{y}, \bar{\omega}, a)$ has parametrisation

$$\left(a' + \frac{a_n - t}{a_n - s}(A + (a_n + t)B)(A + (a_n + s)B)^{-1}[\bar{\omega} - a' - 2s(A + sB)\bar{y}] \right)_t \quad (5.9)$$

which is well defined because the point is not on the fixed curve so $s \neq a_n$, and the matrix $A + (a_n + s)B$ is not singular by Hörmander's criterion (2.2') about non-degenerate critical points. Despite appearances this does give a plane in the case $B = 0$, namely $(\tau_t^{a'})$, although the parametrisation above does not cover the whole of it.

Using this, define a new maximal operator M_δ as follows:

$$M_\delta f(y) = \sup_{\{\omega, a: \delta < |\omega - a'| < 1\}} \frac{1}{|\text{nb}\delta_\delta(\Sigma(y, \omega, a))|} \int_{\text{nb}\delta_\delta(\Sigma(y, \omega, a))} |f(x)| dx.$$

In order to prove the 7/3 bound, we would like to know the following:

Question 3. *For which matrices A and B can we obtain the estimate*

$$\|M_\delta f\|_2 \lesssim r^{-1/2} \delta^{-\varepsilon} \|f\|_2$$

for all f satisfying $\text{supp}(f) \subseteq \text{Ball}(0, 2r) \setminus \text{Ball}(0, r)$ with $0 < r < 1$?

If this were understood (see discussion below), we would be able to obtain the $L^{7/3}$ estimate in three dimensions for all such curves as we now show.

Theorem 28. *If the curves of the form (2.7) with matrices A, B do allow the property of Question 3, then we have an estimate*

$$\|\mathcal{K}_\delta f\|_{L^{q,\infty}(\mathbb{R}^2)} \leq C_\varepsilon \delta^{-(3/p-1+\varepsilon)} \|f\|_{L^{p,1}(\mathbb{R}^3)}$$

with $p = 7/3$ and $q = 7/2$.

This p, q and power of δ lie on the sharp line of Figure 1.3.

Proof: Let $E \subseteq \mathbb{R}^3$ and $\lambda > 0$ be given. Set $\Omega = \{y \in \mathbb{R}^2 : \mathcal{K}_\delta \mathbb{1}_E(y) > \lambda\}$. So we want to show that $|\Omega| \lesssim \lambda^{-7/2} \delta^{-(1+\varepsilon)} |E|^{3/2}$. Let \mathcal{Y} be a maximal ε -separated subset of Ω . Then $\varepsilon^2 \# \mathcal{Y} \geq |\Omega|$, and for each $y \in \mathcal{Y}$ there is a δ -tube T_y such that $|T_y \cap E| > \lambda |T_y|$. So

$$\int_E \sum_{y \in \mathcal{Y}} \mathbb{1}_{T_y} = \sum_{y \in \mathcal{Y}} |T_y \cap E| > \lambda \delta^2 \frac{|\Omega|}{\varepsilon^2}$$

and hence there exists a point $x_0 \in E$ belonging to at least $\frac{\lambda\delta^2|\Omega|}{\varepsilon^2|E|}$ of the tubes. That is, we can find $\mathcal{Y}_0 \subseteq \mathcal{Y}$ with $x_0 \in T_y$ for all $y \in \mathcal{Y}_0$ and $\#\mathcal{Y}_0 \geq \frac{\lambda\delta^2|\Omega|}{\varepsilon^2|E|}$. Define the first bush $B_0 = \bigcup_{y \in \mathcal{Y}_0} T_y$.

Now recall from Lemma 20 that the diameter of the intersection of two ε -separated δ -tubes is at most $C\frac{\delta}{\varepsilon}$. So if we choose ε so that $C\frac{\delta}{\varepsilon} = \lambda/10$ and remove a ball centred at x_0 of this radius, then the remainder of the tubes will be disjoint and still have large intersection with E . More precisely, for all $y \in \mathcal{Y}_0$ we have

$$\begin{aligned} |E \cap (T_y \setminus \text{Ball}(x_0, C\delta/\varepsilon))| &> \frac{9}{10}\lambda|T_y| \\ |B_0 \cap E| &> \frac{9}{10}\lambda \sum_{y \in \mathcal{Y}_0} |T_y| > \frac{9}{10}\lambda|B_0| \end{aligned}$$

Next define $E_1 = E \setminus B_0$ and $\Omega_1 = \{y \in \mathbb{R}^2 : \mathcal{K}_\delta \mathbb{1}_{E_1}(y) > \lambda/2\}$. If $|\Omega_1| \leq |\Omega|/5$, stop. If not, repeat the above construction. We obtain a sequence of bushes with the following properties:

- $B_k = \bigcup_{y \in \mathcal{Y}_k} T_y$
- $\mathcal{Y}_k \subset \Omega_k$ is maximal ε -separated where $C\delta/\varepsilon = \lambda/10$
- $E_k = E \setminus (B_0 \cup \dots \cup B_{k-1})$
- $x_k \in E \cap T_y$ for all $y \in \mathcal{Y}_k$
- $\Omega_k = \{y \in \mathbb{R}^2 : \mathcal{K}_\delta \mathbb{1}_{E_k}(y) > \lambda/2\}$
- $\#\mathcal{Y}_k \geq \frac{\lambda\delta^2|\Omega_k|}{\varepsilon^2|E_k|} \gtrsim \lambda^3 \frac{|\Omega|}{|E|}$
- $|E_k \cap B_k| \geq \frac{9}{10}\lambda|B_k|$

The process terminates after s steps, where $|\Omega_{s+1}| \leq |\Omega|/5$. We require an upper bound for s . Now $\sum_{k=0}^s |E_k \cap B_k| \leq |E|$ since the sets in the sum are disjoint. Hence

$$\begin{aligned} \frac{|E|}{\lambda} &\geq \frac{1}{\lambda} \sum_{k=0}^s |E_k \cap B_k| \\ &\geq \frac{9}{10} \sum_{k=0}^s |B_k| \\ &\geq \frac{9}{10} \sum_{k=0}^s \#\mathcal{Y}_k \delta^2 \\ &\gtrsim s\delta^2 \lambda^3 \frac{|\Omega|}{|E|} \end{aligned}$$

from which it follows that

$$s \lesssim \frac{|E|^2}{\lambda^4 \delta^2 |\Omega|}.$$

Next define $\bar{E} = \bigcup_{k=0}^s (E \cap B_k) = E \setminus E_{s+1}$. Then

$$\{\mathcal{K}_\delta \mathbb{1}_E > \lambda\} \subseteq \{\mathcal{K}_\delta \mathbb{1}_{E_{s+1}} > \lambda/2\} \cup \{\mathcal{K}_\delta \mathbb{1}_{\bar{E}} > \lambda/2\}.$$

Writing this as $\Omega \subseteq \Omega_{s+1} \cup \bar{\Omega}$ we find that $|\bar{\Omega}| > \frac{4}{5}|\Omega|$. For each $y \in \bar{\Omega}$ let T_y be the obvious δ -tube in direction y , so

$$\frac{\lambda}{2}|T_y| < |T_y \cap \bar{E}| = \left| T_y \cap \bigcup_{k=0}^s (E \cap B_k) \right| \leq \sum_{k=0}^s |T_y \cap B_k|. \quad (5.10)$$

Fix k and consider the δ -neighbourhood of the surface consisting of all curves through the point x_k that meet the curve of T_y , that is $\text{nb}_\delta(\Sigma(y, \omega, x_k))$ for some ω . Then it seems plausible that the bush B_k meets a similar proportion of $\text{nb}_\delta(\Sigma(y, \omega, x_k))$ as of T_y , that is,

$$\frac{|\text{nb}_\delta(\Sigma(y, \omega, x_k)) \cap B_k|}{|\text{nb}_\delta(\Sigma(y, \omega, x_k))|} \sim \frac{|T_y \cap B_k|}{|T_y|} \quad (5.11)$$

although we should check that all curves in the bush that do not meet T_y can meet the surface $\Sigma(y)$ only in a proper intersection (that is, not sharing a tangent). Now using the property about the maximal function over such surfaces, and denoting the “truncated bush” $(B_k - x_k) \cap (\text{Ball}(0, 2r) \setminus \text{Ball}(0, r))$ by B_k^r we have by (5.10) and (5.11)

$$\lambda \lesssim \sum_{k=0}^s \sum_{l=0}^{\log \frac{1}{\delta}} M_\delta \mathbb{1}_{B_k^{\delta 2^l}}(y).$$

Apply Cauchy-Schwarz to get

$$\lambda^2 \lesssim s \log \frac{1}{\delta} \sum_{k=0}^s \sum_{l=0}^{\log \frac{1}{\delta}} M_\delta \mathbb{1}_{B_k^{\delta 2^l}}(y)^2$$

and then integrate over all $y \in \bar{\Omega}$, applying the L^2 estimate for M to get

$$\lambda^2 |\bar{\Omega}| \lesssim s \delta^{-\epsilon} \sum_{k=0}^s \sum_{l=0}^{\log \frac{1}{\delta}} \frac{1}{\delta^{2^l}} |B_k^{\delta 2^l}|.$$

By the geometry of the truncated bushes, $|B_k^r| \lesssim r |B_k|$ so that we finally obtain

$$\begin{aligned} \frac{9}{10} \lambda^2 |\Omega| &\leq \lambda^2 |\bar{\Omega}| \lesssim s \delta^{-\epsilon} \sum_{k=0}^s \sum_{l=0}^{\log \frac{1}{\delta}} |B_k| \\ &\lesssim \frac{|E|^2}{\lambda^4 \delta^2 |\Omega|} \delta^{-\epsilon} \frac{|E|}{\lambda} \end{aligned}$$

using the estimates above, which rearranges to $|\Omega|^2 \lesssim \frac{|E|^3}{\delta^{2+\epsilon}\lambda^7}$, as required. \square

So, how might we prove the estimate of Question 3? In the plane case the proof is short, and uses Córdoba's L^2 argument. So M_δ is written in terms of the Kakeya maximal function in \mathbb{R}^2 as follows:

$$\frac{1}{|\text{nbd}_\delta(\Sigma(y))|} \int_{\text{nbd}_\delta(\Sigma(y))} |f| dx \lesssim \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathcal{K}_\delta(f|_{L+te_L})(\tilde{y}) dt$$

where L is a plane with normal e_L including the straight line Γ_y , the vector $\tilde{y} \in \mathbb{B}^1$ describes the direction of Γ_y in this two-plane and ε depends on the orientation of $\text{nbd}_\delta(\Sigma(y))$ (a parallelepiped) with respect to L . This is illustrated in Figure 5.3. It seems difficult to translate this into the curved case in an appropriate way.

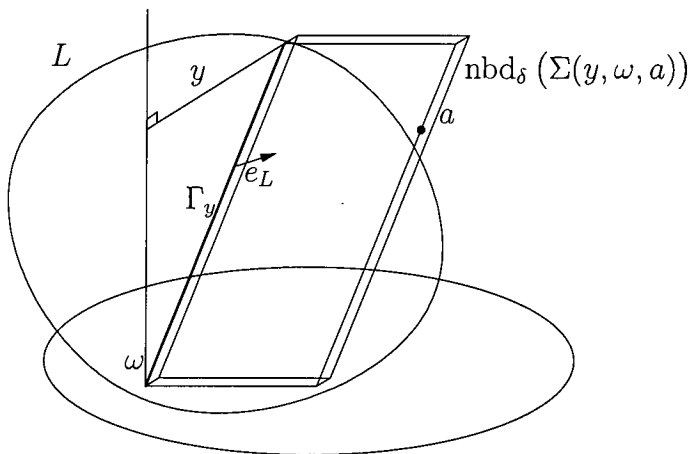


Figure 5.3: *Proving the M_δ estimate in the straight line case*

If we let L be a flat plane (there is only one that includes Γ_y), then we shall probably find that as we vary the height of L , its curve of intersection with the surface $\Sigma(y)$ will no longer belong to our usual family of curves, and will depend on t . We could instead let L be another of the surfaces Σ_y , but this seems even more complicated.

Anyway, good behaviour of M_δ would seem to require that surfaces intersect properly, so that their δ -neighbourhoods intersect in a δ -tube rather than something bigger as would happen if the surfaces shared a tangent plane. We shall notice similar things in the next section.

5.4 Wolff's approach

The $\frac{n+2}{2}$ result in the straight line case was originally due to Wolff [40]. His result was slightly stronger than that given by Katz's simple proof which we looked at in Section 5.2, since Wolff gave a sharp bound from $L^{\frac{n+2}{2}}$ while Katz's result

is what is obtained from this by interpolation (although Katz also proved an $L^{\frac{n+2}{2}} \rightarrow L^{\frac{n+2}{2}}$ result in an appendix, which we did not discuss in Section 5.2). Both are sufficient to give the lower bound for the set dimension. Also, by an axiomatic approach Wolff managed to cover both the Kakeya and Nikodym cases at the same time.

This paper was the first to use “hairbrushes”, and like Katz’s version the proof has two parts; one showing that hairbrushes behave well, and another showing that in every collection of tubes one can find enough hairbrushes so that the remaining tubes are almost disjoint. However, Wolff’s method for the first part is very different from that of Katz, and in the Kakeya version runs as follows:

Given a hairbrush—that is, a tube with a large number of other tubes meeting it—consider the family of planes through the central tube. Each plane includes the family of all lines that meet the central one whose direction belongs to a one-parameter family, as shown in Figure 5.4. If we consider neighbourhoods of

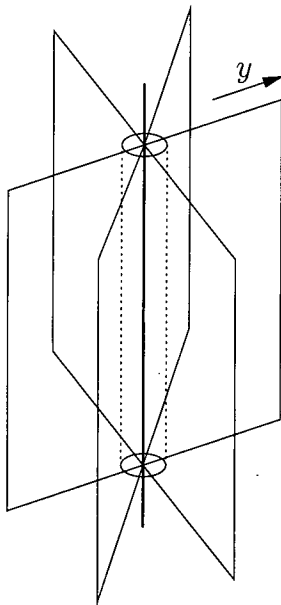


Figure 5.4: Wolff’s “foliation” idea: All lines whose direction is given by a multiple of y lie in the plane marked

the planes, finitely many of these will include all the δ -tubes in the hairbrush, while being disjoint a suitable distance away from the central tube. Meanwhile in each plane, we can apply the \mathbb{R}^2 result of Córdoba to show that the tubes are almost disjoint. Although mixing metaphors somewhat, this idea of grouping the tubes into planes has been termed “foliation”.

Of course this description is rather imprecise, but will be sufficient to illustrate the barrier encountered in generalising to curves.

The first problem is that two given curves need not lie in a common plane.

Being parabolas, the curves of the form (2.7) are of course planar, but pairs of them need not share the same plane. Let us suppose that the central curve of our hairbrush is $\Gamma_0(0)$, the x_n -axis. Then we would require that the other curve $\Gamma_{y_0}(\omega_0)$ lies in a *vertical* plane, which is true if and only if Ay_0 and By_0 are parallel.

Instead of grouping the curves into planes, we might try surfaces. One can easily check that Córdoba's argument works in a surface provided that, as here, the curves intersect properly, and that the directions of the curves lying in a given surface may be described using a single parameter.

One possible generalisation of the planes is to consider the locus of all translates of the curve in direction y_0 that meet the given central curve $\Gamma_0(0)$. If we denote the height where they intersect by s , then it follows that $\omega = 2s(A + sB)y_0$, and hence the parametrisation of this surface, which we shall call S_{y_0} , is

$$\begin{pmatrix} 2(s-t)(A + (s+t)B)y_0 \\ t \end{pmatrix}. \quad (5.12)$$

Clearly if $B = 0$, or if $By = Ay$, then this is a plane as expected. By analogy with the argument above, we would like this surface to include curves in the one-parameter family of directions αy_0 , $\alpha \in \mathbb{R}$. Unfortunately this is false in general. To see this, let the surface for y_0 be parametrised as above, and consider the surface $S_{\alpha y_0}$, which has parametrisation $\begin{pmatrix} 2(u-t)(A + (u+t)B)\alpha y_0 \\ t \end{pmatrix}$. If these coincide, then by substituting $t = 0$ we find that $2(s - u\alpha)Ay \equiv 2(s^2 - u^2\alpha)By$ for all s , where $u = u(s)$. Unless Ay is parallel to By this can happen only if $s = u\alpha$ and $\alpha = 1$.

We might try to salvage this by using a separate surface for each $y \in \mathbb{B}^{n-1}$: $Ay \nparallel By$, instead of $y \in S^{n-2}$ as in the straight line case. However, not only is this likely to lead to a much weaker result (perhaps just the trivial $\frac{n+1}{2}$ bound), but we have the additional problem that these surfaces meet tangentially. The tangent plane to the surface (5.12) is spanned by

$$\begin{pmatrix} -2(A + 2tB)y_0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2(A + 2sB)y_0 \\ 0 \end{pmatrix}$$

and so along the x_n -axis where S_{y_0} and $S_{\alpha y_0}$ meet, their tangent vectors are parallel. This means that we cannot hope for the δ -neighbourhoods of two surfaces whose directions y are d -separated to be disjoint at a distance of the order of δ/d from the central curve, but rather $\sqrt{\delta}/d$ (cf. Lemma 20, Figure 4.1). Thinking of our negative results of Chapter 4 it seems clear that this property would prevent us from proving the maximal function result by this method, although the Minkowski result (where surfaces are only required to be disjoint at a distance of an absolute constant) might still be possible.

Worse still, though, two surfaces can meet at points arbitrarily far away from the central curve. The extreme of this occurs where $BA^{-1}B = 0$, where the surfaces can meet along another whole line in addition to the x_n -axis, as illustrated in Figure 5.5. However, the matrices in Bourgain's worst case example have this

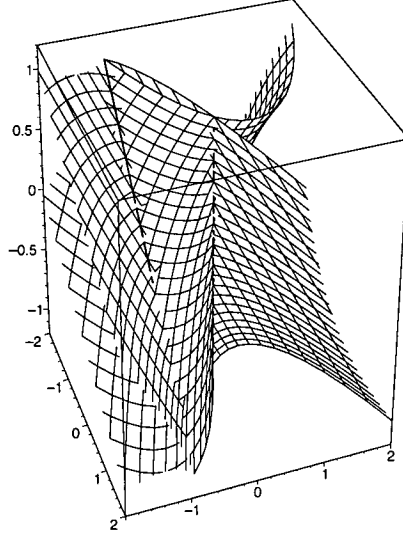


Figure 5.5: *Surfaces can meet along two lines in the case $BA^{-1}B = 0$*

property, so as shown on page 51 we would not expect to be able to obtain the $\frac{n+2}{2}$ result anyway. The property $BA^{-1}B = 0$ is discussed further in Chapter 7.

With general matrices, two such surfaces through the x_n -axis may not meet along another curve, but they might meet at points well away from the axis. For if we fix s and t then the point $x = (x', t)$ with $x' := (s - t)(A + 2(s + t)B)y_0$ lies not only on the surface S_{y_0} , but on every S_y with y of the form

$$\frac{1}{u - t}(A + 2(u + t)B)^{-1}x'$$

for some u . This finally removes all hope of imitating Wolff's proof in this way.

Of course, it may be that our choice of surface is wrong: There are many ways to define a surface passing through two given curves that give a plane in the straight line case. S_{y_0} was constructed by fixing the direction y_0 and considering the locus of all curves $\Gamma_{y_0}(\omega)$ where ω was allowed to vary in such a way that the curve always intersected $\Gamma_0(0)$. Instead, we could create a surface consisting of all curves of the form $\Gamma_{\alpha y_0} : \alpha \in \mathbb{R}$ that meet $\Gamma_0(0)$ at the same point as the original curve, thereby removing at least the first of the difficulties mentioned above. This surface has parametrisation

$$\begin{pmatrix} \alpha\omega_0 - 2\alpha t(A + tB)y_0 \\ t \end{pmatrix}$$

which geometrically is just the surface generated by joining the two curves by a horizontal straight line at every height. This might seem promising, since it does

include curves in a one-parameter family of directions, but if we try to include *all* curves in these directions that meet the axis, we are back to considering $S_{\alpha y_0}$ again. What is more, it is a simple matter to find two *intersecting* curves that both meet the axis, and then their corresponding surfaces are different and yet meet along an entire horizontal line. More precisely, for any curve $\Gamma_{y_0}(\omega_0)$, the curve $\Gamma_z(0)$ where $z = y_0 - (A + \frac{1}{2}B)^{-1}\omega_0$ will do, where the $\frac{1}{2}$ appears because this example has the surfaces coinciding at height $t = 1/2$. An illustration is given in Figure 5.6.

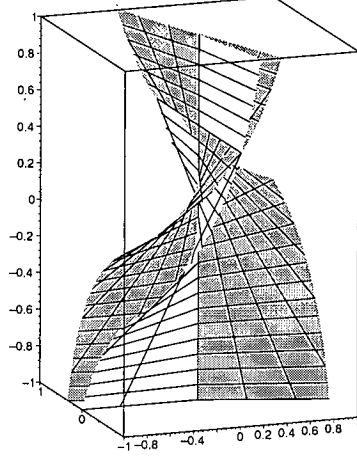


Figure 5.6: *Two surfaces meeting along an entire horizontal line*

Thus we have not managed to find any way of imitating the foliation idea for curves, since no definition of surface seems to have all the properties we require. Sadly the recent works of Katz, Łaba and Tao [20, 24] also build on this idea, so there is reason for pessimism about adapting those for curves too. Fortunately we have found greater success with the arithmetic methods, which we turn to now.

Chapter 6

Arithmetic Methods

*While I was in the dentist's chair I thought about the Katz-Tao Lemma.
Tony Carbery*

6.1 Introduction

Chapters 4 and 5 showed how geometric methods could give lower bounds for the set dimension of the form $\frac{n}{2} + \text{const}$. Such bounds are almost the best known in low dimensions ($n = 3$ or 4), but in higher dimensions far better results are obtained by an arithmetic approach, since these improve the coefficient of n to something greater than $1/2$.

The arithmetic arises in the form of *sumset inequalities*. For these we require some notation.

Notation. Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{Z}^{n-1}$ be finite sets and let $\mathcal{G} \subseteq \mathcal{A} \times \mathcal{B}$. For any $(n-1) \times (n-1)$ real matrix X define the X -sumset of \mathcal{A} and \mathcal{B} by

$$\mathcal{A} + X\mathcal{B} := \{a + Xb : (a, b) \in \mathcal{G}\}.$$

In the case $X = -I$ write $\mathcal{A} - \mathcal{B}$ and call it the difference set.

The structure of sumsets, and inequalities regarding the relative sizes of sum and difference sets, have been extensively studied by combinatorialists when the matrix X is an integer multiple of the identity, but they have generally considered only $\mathcal{G} = \mathcal{A} \times \mathcal{B}$. See [26, 29]. The link with the Kakeya problem was noticed in 1999 by Bourgain [5], and since then many inequalities with $\mathcal{G} \subseteq \mathcal{A} \times \mathcal{B}$ have been proved. However, the case where X is not a multiple of I arises only with curves, and seems to be a new problem.

6.1.1 The straight line case

We begin by looking at the straight line case. Bourgain's idea is summed up in the following lemma¹.

Lemma 29 ([5]). *Suppose that there exists an $\varepsilon > 0$ such that for all $\mathcal{G} \subseteq \mathcal{A} \times \mathcal{B}$ an a priori estimate*

$$\#(\mathcal{A} - \mathcal{B}) \leq \max\{\#\mathcal{A}, \#\mathcal{B}, \#(\mathcal{A} + \mathcal{B})\}^{2-\varepsilon}$$

for the difference set in terms of the sets and their sumset with $X = I$ holds. Then (straight) Kakeya sets have Minkowski dimension at least $\frac{n-1}{2-\varepsilon} + 1$.

Of course such an estimate with $\varepsilon = 0$ is obvious, and recovers the “trivial bound” discussed in Chapter 4. At the other extreme $\varepsilon = 1$ (which is unfortunately not possible in the inequality above) would prove the Kakeya conjecture.

Proof: Let E be a Kakeya set and imagine two horizontal planes through $\text{nb}\delta(E)$. As before we may assume that this consists of finitely many tubes in δ -separated directions. Let \mathcal{A} and \mathcal{B} be the intersections of $\text{nb}\delta(E)$ with each of the planes—because of the δ -discretisation, we may assume (see [41]) that \mathcal{A} and \mathcal{B} are subsets of $\delta\mathbb{Z}^n$. Since two points determine a straight line, we are interested in the set $\mathcal{G} \subseteq \mathcal{A} \times \mathcal{B}$ given by

$$\mathcal{G} = \{(a, b) : a \text{ and } b \text{ lie on the same tube in } \text{nb}\delta(E)\}.$$

The sumset $\mathcal{A} + \mathcal{B}$ then corresponds to the intersection of $\text{nb}\delta(E)$ with a plane halfway between the first two (scaled by a factor of two, which does not change the cardinality). Now by the assumption, the difference set has cardinality at most $\max\{\#\mathcal{A}, \#\mathcal{B}, \#(\mathcal{A} + \mathcal{B})\}^{2-\varepsilon}$. But since we assumed that the directions $a - b$ were all distinct, the difference set is the same size as \mathcal{G} itself, that is, about $C\delta^{-(n-1)}$. So one of the three sets has cardinality at least $\delta^{-\frac{n-1}{2-\varepsilon}}$. If we now vary the heights of \mathcal{A} , \mathcal{B} and hence of the slice halfway between, we always have at least one of the intersections having large cardinality. So one of the three, \mathcal{A} say, has this property for a range of heights of positive measure. This means that small cylinders (of width δ and height a small constant c) are included in $\text{nb}\delta(E)$. This gives

$$\begin{aligned} |\text{nb}\delta(E)| &\geq C\delta^{n-1}\#\mathcal{A} \times c \\ &\geq C\delta^{n-1-\frac{n-1}{2-\varepsilon}} \end{aligned} \tag{6.1}$$

¹In fact, Bourgain stated his result without the extra +1 in the lower bound; this improvement was noticed by Katz and Tao [21]. Compare with Lemma 32.

which shows that $\dim(E) \geq \frac{n-1}{2-\varepsilon} + 1$ as required. \square

So the point is that if a set has small dimension, then all horizontal slices through its δ -neighbourhood are small, which means that the sets of points of intersection and their (positive, scalar) sumsets are small; yet if it is a Kakeya set then it includes lines in all directions which forces the difference set to be large.

To get a positive ε , these properties of the finite sets must fight against one another. This is reasonable, since the assumption that the sumset has cardinality roughly that of the original sets (rather than their product), is saying something about the structure of those sets, which in turn suggests that the difference set will also be smaller than the trivial estimate of $\#\mathcal{A}\#\mathcal{B}$.

In his paper, Bourgain obtains a suitable but weak sumset inequality ($\varepsilon = \frac{1}{13}$) using some rather complicated ideas related to Gowers' version of the Balog-Szemerédi theorem. Roughly, this attempts to describe sets with small sumset in terms of having large intersection with arithmetic progressions. However, the best estimates known are due to Katz and Tao and are based on the trivial observation that if two sums coincide, that is $a + b = a' + b'$ with $(a, b), (a', b') \in \mathcal{G}$, then the differences $a - b'$ and $a' - b$ also coincide. The difficulty is that (a, b') and (a', b) may not be in \mathcal{G} . Their earliest result of this type is the following:

Lemma 30 ([21]). *With the notation above, we have*

$$\#(\mathcal{A} - \mathcal{B}) \leq \max \{ \#\mathcal{A}, \#\mathcal{B}, \#(\mathcal{A} + \mathcal{B}) \}^{2-1/6}$$

for all finite $\mathcal{A}, \mathcal{B} \subseteq \mathbb{Z}^n$ and $\mathcal{G} \subseteq \mathcal{A} \times \mathcal{B}$.

Combining this with Bourgain's result (Lemma 29) shows that Kakeya sets have Minkowski dimension at least $\frac{6n+5}{11}$.

Of course, there is no reason why we must consider only the plane halfway between the first two, nor are we restricted to using only three "slices". Other planes simply correspond to different sumsets such as $\mathcal{A} + 2\mathcal{B}$, and adding more such assumptions gives better bounds for the difference set. These ideas have been pursued by Katz and Tao [23], leading to the lower bound of approximately $0.5969n + 0.403$ for the dimension of Kakeya sets, which is currently the best known in high dimensions. They have also found ways to extend the technique to the harder Hausdorff dimension and maximal function problems, but we do not do so here.

6.2 Formulation of the problem for curves

In this section we show how the same technique of taking slices through a curved Kakeya set leads to sumset problems that involve matrices. We shall also do this

for the Nikodym problem. The idea is illustrated in Figure 6.1. The main issue is

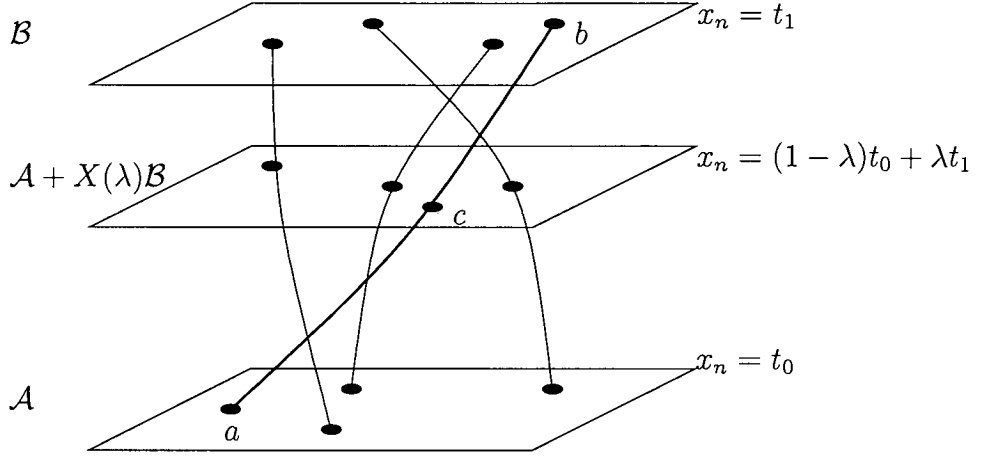


Figure 6.1: Slices through a curved Kakeya set

that if (a, t_0) and (b, t_1) lie on a curved tube, then the point of intersection of the curve with the plane at height $\frac{t_0+t_1}{2}$ need not be $(\frac{a+b}{2}, \frac{t_0+t_1}{2})$ —this is why matrices are needed. Also, is it the case that the “direction” of the curve is uniquely determined by $a - b$?

Lemma 31. *Let $\mathcal{A}, \mathcal{B} \subset \mathbb{Z}^{n-1}$ be the $(\delta$ -discretised) intersections of a set E of curves of the form (2.7) with the planes $x_n = t_0$ and $x_n = t_1$ respectively ($t_0 \neq t_1$). Let*

$$\mathcal{G} := \{(a, b) : a \text{ and } b \text{ lie on the same tube in } \text{nb}\delta_\delta(E)\} \subseteq \mathcal{A} \times \mathcal{B}.$$

Then

- *The set of directions y has the same cardinality as the difference set $\mathcal{A} - \mathcal{B}$.*
- *Assume that $t_0, t_1 \neq 0$. The set of centres ω has the same cardinality as $\mathcal{A} - T\mathcal{B}$ where T is the $(n-1) \times (n-1)$ matrix*

$$T = \frac{t_0}{t_1}(A + t_0B)(A + t_1B)^{-1}.$$

- *The intersection of the set with the plane $x_n = (1 - \lambda)t_0 + \lambda t_1$ has the same cardinality as the sumset $\mathcal{A} + X(\lambda)\mathcal{B}$, where $X(\lambda)$ is the $(n-1) \times (n-1)$ matrix*

$$X(\lambda) = \frac{\lambda}{1 - \lambda} \left[I + \lambda(t_1 - t_0)B(A + (t_0 + t_1)B)^{-1} \right]^{-1} \left[I - (1 - \lambda)(t_1 - t_0)B(A + (t_0 + t_1)B)^{-1} \right].$$

Proof: Consider a curve through the points (a, t_0) and (b, t_1) . The equation (2.7) of the curves gives

$$a = \omega - 2t_0Ay - 2t_0^2By \quad (6.2)$$

$$b = \omega - 2t_1Ay - 2t_1^2By. \quad (6.3)$$

Subtracting these we find that $y = \frac{1}{2(t_0 - t_1)}(A + (t_0 + t_1)B)^{-1}(b - a)$, and so the first assertion follows, since multiplication by an invertible matrix does not change the cardinality.

If we write $M_j = 2t_j(A + t_jB)$ for $j = 0, 1$ so that $a = \omega - M_0y$, $b = \omega - M_1y$ then solving gives

$$\omega = (M_0^{-1} - M_1^{-1})^{-1}(M_0^{-1}a - M_1^{-1}b).$$

We can always multiply through by an invertible matrix to get the a on its own. Therefore an appropriate “difference set” is $\mathcal{A} - T\mathcal{B}$ where $T = M_0M_1^{-1} = \frac{t_0}{t_1}(A + t_0B)(A + t_1B)^{-1}$ as in the second assertion. Note that for the Nikodym problem, we cannot take slices through $x_n = 0$ anyway, because by our definition, a neighbourhood of this plane has been deleted from the Nikodym set.

For the third, denote the point of intersection of this curve with the intermediate plane by c . It helps to take $(1 - \lambda)(6.2) + \lambda(6.3)$, which gives

$$\omega = (1 - \lambda)a + \lambda b + (1 - \lambda)(2t_0Ay + 2t_0^2By) + \lambda(2t_1Ay + 2t_1^2By).$$

This allows lots of cancellation, so that

$$\begin{aligned} c &= \omega - 2((1 - \lambda)t_0 + \lambda t_1)Ay - 2((1 - \lambda)t_0 + \lambda t_1)^2By \\ &= (1 - \lambda)a + \lambda b + 2\lambda(1 - \lambda)(t_0 - t_1)^2By \\ &= (1 - \lambda)a + \lambda b + 2\lambda(1 - \lambda)(t_0 - t_1)^2B \frac{1}{2(t_0 - t_1)}(A + (t_0 + t_1)B)^{-1}(b - a) \\ &= \left[(1 - \lambda)I + (1 - \lambda)\lambda(t_1 - t_0)B(A + (t_0 + t_1)B)^{-1} \right] a + \\ &\quad + \left[\lambda I - \lambda(1 - \lambda)(t_1 - t_0)B(A + (t_0 + t_1)B)^{-1} \right] b. \end{aligned}$$

Multiplying through by an invertible matrix gives the result. \square

It is easy to check that all the matrices occurring above are indeed invertible, because of the non-degeneracy criterion (2.2'). Recall also that in the straight line case we have $B = 0$ and hence $X(\lambda)$ is really just a scalar. The sumset in the second assertion does not appear in the literature on the straight line problem since it is only appropriate when dealing with Nikodym rather than Kakeya sets. Although the elements of the matrices A, B and the parameters

t_0, t_1, λ are real, for our application we should consider only matrices over \mathbb{Q} , since each real number may be approximated to within $O(\delta)$ by a rational, which corresponds to the same point in the δ -discretisation.

To summarise, the most general matrix sumset problem with $N + 2$ “slices” is as follows:

Question 4. *Let X_1, X_2, \dots, X_N be $n \times n$ real matrices. To avoid trivialities assume that they are non-zero, distinct, and not equal to $-I$. Does there exist an $\varepsilon > 0$ depending only on the X_j s such that for all $\mathcal{A}, \mathcal{B} \in \mathbb{Z}^n, \mathcal{G} \subseteq \mathcal{A} \times \mathcal{B}$ we have*

$$\#(\mathcal{A} - \mathcal{B}) \leq \max \left\{ \#\mathcal{A}, \#\mathcal{B}, \max_j \#(\mathcal{A} + X_j \mathcal{B}) \right\}^{2-\varepsilon} ?$$

If so, what is the largest possible ε ?

This applies to curved Kakeya and Nikodym sets of parabolas via the obvious generalisation of Lemma 29.

Lemma 32. • *Suppose that for some coefficients A, B we can choose $t_0, t_1 \in [-1, 1]$ and $\lambda_j \in (0, 1)$, $j = 1, \dots, N$ such that Question 4 with $X_j = X(\lambda_j)$ has a positive answer. Then Kakeya sets of curves of the form (2.7) for this A, B have Minkowski dimension at least $\frac{n-1}{2-\varepsilon}$.*

- *If the same holds but with $X_j = X(\lambda_j)T^{-1}$ where T is as in Lemma 31 and none of the heights $t_0, t_1, (1 - \lambda_j)t_0 + \lambda_j t_1$ is 0, then the corresponding curved Nikodym sets have Minkowski dimension at least $\frac{n-1}{2-\varepsilon}$.*
- *In both cases, if in fact we have a range of solutions, meaning that as t_0 is allowed to vary over some small interval and the other heights to vary correspondingly, then we can obtain the better lower bound of $\frac{n-1}{2-\varepsilon} + 1$.*

Proof: The proof is exactly as in Lemma 29, except that we take $N + 2$ slices through the set, at heights t_0, t_1 and $(1 - \lambda_j)t_0 + \lambda_j t_1$, $j = 1, \dots, N$. Define \mathcal{G} as before. For the Nikodym version redefine $\mathcal{B}' = T\mathcal{B}$, which does not change the cardinality; but for Kakeya let $\mathcal{B}' = \mathcal{B}$. Then by δ -separatedness of centres or directions respectively, we find that $\#\mathcal{G} = \#(\mathcal{A} - \mathcal{B}') = \delta^{-(n-1)}$. By the sumset inequality assumed, this implies that either \mathcal{A} or \mathcal{B}' or one of their X_j -sumsets must have large cardinality. But in either case, $\mathcal{A} + X_j \mathcal{B}' = \mathcal{A} + X(\lambda_j)\mathcal{B}$ so this means we have found a slice through the set with large measure.

If we can vary the heights of the planes, then we obtain the lower bound of $\frac{n-1}{2-\varepsilon} + 1$ as in Lemma 29. If not, then instead of cylinders, we merely deduce that the set includes δ -balls at the intersections with the largest slice. So we must replace c by δ in equation (6.1) and hence we deduce only that the dimension is at least $\frac{n-1}{2-\varepsilon}$. \square

6.3 Sumsets in general

In this section we discuss Question 4 away from the context of Kakeya and Nikodym sets. That is, we try to find matrices X_j that will work, without regard to whether they can be realised as $X(\lambda)$ or $X(\lambda)T^{-1}$ for some choice of the parameters in our original problem. The latter will be the subject of Section 6.4.

6.3.1 The scalar case

We begin by reviewing the known results in the case where all of the X_j are multiples of the identity.

With three slices ($N = 1$) and $X_1 = I$ we have the problem Bourgain originally used in [5]. He proved the estimate with $\varepsilon = \frac{1}{13}$, which was quickly improved to $\frac{1}{6}$ by Katz and Tao [21]. For all other rational multiples of I the existence of positive improvements ε has been proved by Christ [10], although it is tedious to compute their values.

The first four-slice estimate was again due to Katz and Tao, namely

$$\#(\mathcal{A} - \mathcal{B}) \leq \max \{ \# \mathcal{A}, \# \mathcal{B}, \#(\mathcal{A} + \mathcal{B}), \#(\mathcal{A} + 2\mathcal{B}) \}^{2^{-1/4}} \quad (6.4)$$

as shown in [21]. In [23, Theorem 3.3] they showed that $\varepsilon = 1/4$ still holds if, instead of using 1 and 2 as here, the two non-zero scalars simply differ by 1. We shall generalise this for matrices shortly.

In the same theorem, they showed that six slices with scalars x, y, \bar{x}, \bar{y} satisfying

$$(1 + \frac{1}{x})x = (1 + \frac{1}{y})y \quad (6.5)$$

also gave $\varepsilon = 1/4$. This relation allows us to obtain results for five slices also, by taking two scalars to be equal (or by taking one to be ∞ , in which case we interpret $\mathcal{A} + \infty\mathcal{B} =: \mathcal{B}$).

They also proved an iteration result:

Theorem 33 ([23]). *If we can obtain $\varepsilon = \varepsilon_0$ in Question 4 for some finite set of scalars, then for some larger set of scalars we can obtain $\varepsilon = \frac{2-\varepsilon_0^2}{8-7\varepsilon_0+\varepsilon_0^2}$. Hence by choosing larger and larger sets, the improvement ε may be made as close to the fixed point $0.32486\dots$ as we wish.*

This result gives the lower bound of approximately $0.5969n + 0.403$ for the Minkowski dimension of straight-line Kakeya sets, which is currently the best known for large n .

6.3.2 General matrices

Question 4 with the X_j not multiples of I seems hard. However, we have some negative results, and have been able to generalise some of the positive results from the scalar case. We begin with a rather trivial observation.

Lemma 34. *If all of the X_j are block diagonal with blocks of the same size, then a sumset inequality for these X_j implies one for each of the sets of blocks, with the same ε .*

Proof: Obvious by letting \mathcal{A}, \mathcal{B} consist of vectors with zeros everywhere except in the block of interest. \square

The converse seems likely to be false—we would need not only that “collisions” often occur in each block of coordinates, but that they often occur in all coordinates at the same time.

We now reveal the easy but disappointing fact that three slices is simply not enough in the matrix case.

Theorem 35. *If X is not a multiple of the identity, then the power of 2 in*

$$\#(\mathcal{A} - \mathcal{B}) \leq \max \{ \# \mathcal{A}, \# \mathcal{B}, \#(\mathcal{A} + X\mathcal{B}) \}^2$$

is best possible.

Proof: Choose a vector v that is not an eigenvector of X , and let \mathcal{B} consist of M equally spaced points along this direction. Set $\mathcal{A} = \{Xb : b \in \mathcal{B}\}$, that is, M equally spaced points along the direction Xv . Then with $\mathcal{G} = \mathcal{A} \times \mathcal{B}$, clearly $\#(\mathcal{A} + X\mathcal{B})$ is about $2M$, while since v and Xv are linearly independent, $\#(\mathcal{A} - \mathcal{B})$ is about M^2 . \square

Similar observations with more slices give another negative result.

Theorem 36. *Suppose that there exists $v \in \mathbb{R}^n$ such that all the vectors $X_j v$ are rational multiples of some fixed vector w which is not parallel to v itself. (That is, v is a secular vector of each pair of matrices, but is not an eigenvector.) Then there can be no positive answer to Question 4.*

This theorem is rather weak, but it does at least rule out the case where the matrices X_j are all multiples of each other but not of the identity, and combining this with Lemma 34 gives further examples. This makes sense because taking more than three slices is not really giving much more information.

Proof: We have $X_j v = \frac{p_j}{q_j} w$ where p_j, q_j are non-zero coprime integers. Let $M > \prod_{i=1}^N q_i$ be a large integer, and set

$$\mathcal{A} = \left\{ n \left(\prod_{j=1}^N p_j q_j \right) w : n = 1, \dots, M \right\}$$

$$\mathcal{B} = \left\{ n \left(\prod_{i=1}^N q_i \right) v : n = 1, \dots, M \right\}.$$

Then if $\mathcal{G} = \mathcal{A} \times \mathcal{B}$, we find that

$$\mathcal{A} + X_j \mathcal{B} = \left\{ p_j \left(\prod_{k \neq j} q_k \right) \left[q_j \left(\prod_{i \neq j} p_i \right) m + n \right] w : m, n = 1, \dots, M \right\}$$

$$\mathcal{A} - \mathcal{B} = \left\{ m \left(\prod_{j=1}^N p_j q_j \right) w - n \left(\prod_{i=1}^N q_i \right) v, n = 1, \dots, M \right\}$$

and we have the combinatorial task of finding their cardinality. Since w and v are linearly independent it is obvious that $\#(\mathcal{A} - \mathcal{B}) = M^2$. To find the cardinality of $\mathcal{A} + X_j \mathcal{B}$, first write $Q_j := q_j \prod_{i \neq j} p_i$. We need to know how many distinct numbers $(mQ_j + n)$ there are, so first let $m < M$ be fixed. Since $M > Q_j$, there are Q_j such numbers between $mQ_j + 1$ and $(m+1)Q_j$ inclusive, after which the remaining values overlap with those for the next m . When $m = M$ we just get all the numbers $MQ_j + 1$ up to $MQ_j + M$. Hence, we find that $\#(\mathcal{A} + X_j \mathcal{B}) = Q_j(M-1) + M$. So \mathcal{A} and \mathcal{B} and all the sumsets all have cardinality about M while the difference set is about M^2 . So there can be no positive answer to Question 4. \square

So far this picture looks bleak. However, we can prove one or two results in the positive direction, analogous to those in the scalar case. Here we generalise the four-slice result (6.4) to the matrix setting. For legibility, write $X_1 = X$, $X_2 = Y$.

Theorem 37. *If $Y - X = I$ and X is invertible, then*

$$\#(\mathcal{A} - \mathcal{B}) \leq \max \{ \# \mathcal{A}, \# \mathcal{B}, \#(\mathcal{A} + X \mathcal{B}), \#(\mathcal{A} + Y \mathcal{B}) \}^{7/4}.$$

Proof: This is just as in [21], so we only give an outline. Start by discarding elements of \mathcal{G} until $\#(\mathcal{A} - \mathcal{B}) = \# \mathcal{G}$, and denote the maximum on the right hand side by M . We need to show that $\# \mathcal{G} \leq M^{7/4}$.

The idea is to count *trapezia*: sets of four elements of \mathcal{G} consisting of two “sides” whose endpoints have the same value of a while the endpoints of the remaining two sides share values of $a + Yb$ and b respectively. More precisely, a trapezium is a set

$$\{(a_0, b_0), (a_0, b'_0), (a_1, b_1), (a_1, b'_1)\} \subseteq \mathcal{G}$$

such that $a_0 + Yb_0 = a_1 + Yb_1$ and $b'_0 = b'_1$.

First count the number of pairs in \mathcal{G} that share their value of a . This is

$$\begin{aligned} \#\{(a, b), (a, b') \in \mathcal{G}\} &= \sum_{a \in \mathcal{A}} \#\{b : (a, b) \in \mathcal{G}\}^2 \\ &\geq \frac{\#\mathcal{G}^2}{\#\mathcal{A}} \end{aligned}$$

by Cauchy-Schwarz. A trapezium consists of two such pairs that share their value of $(a + Yb, b')$, so by Cauchy-Schwarz again we find that the number of trapezia is at least

$$\frac{(\#\mathcal{G}^2/\#\mathcal{A})^2}{\#(\mathcal{A} + Y\mathcal{B})\#\mathcal{B}} \geq \frac{\#\mathcal{G}^4}{M^4}.$$

But we also have the following algebraic fact:

$$a_1 - b'_1 = (I + X^{-1})(a_0 + Xb_0) - X^{-1}(a_0 + Xb'_0) - Yb_1.$$

So, since $\#(\mathcal{A} - \mathcal{B}) = \#\mathcal{G}$, knowing $(a_0 + Xb_0)$, $(a_0 + Xb'_0)$, and b_1 is enough to determine (a_1, b'_1) , and hence the whole trapezium by substituting back. So the number of trapezia is at most M^3 , which together with the lower bound of $\#\mathcal{G}^4/M^4$ gives the result. \square

Study of [23] suggests that five or six slices, where the matrices satisfy relations derived from (6.5) like

$$\begin{aligned} 0 &= Y - X + Z^{-1}Y \\ \text{or } 0 &= Y - X + Z^{-1}Y - W^{-1}X \end{aligned}$$

would give the same $\varepsilon = 1/4$ bound. It may also be possible to imitate Katz and Tao's iteration result, thus obtaining very good sumset bounds for large collections of matrices. However, we have not pursued this here since it is not clear that these bounds could be successfully applied to the set dimension problems, as we shall see in the next section.

6.4 Application to the dimension of sets

We now return to the original problem of finding lower bounds for the Minkowski dimension of curved Kakeya and Nikodym sets. As we shall see, even where the sumset question has a positive answer for particular matrices, it is not at all easy to realise such matrices as those occurring when we take slices through a set. (Recall Lemma 32.) Our plan is to work through the results of the last section applying them to the Kakeya and Nikodym problems. In most cases this turns out to be either unsuccessful or apparently intractable, except for the Nikodym sets that worked in Section 5.2 and some other special cases, where we are able to obtain good bounds.

6.4.1 Scalars

It makes sense to begin by seeing what can be squeezed out of the well known scalar sumset results in the curved case.

Lemma 38. • $X(\lambda)$ cannot be a multiple of the identity except in the straight line case.

- $X(\lambda)T^{-1}$ is a multiple of the identity if $BA^{-1}B = 0$, but this condition is not necessary.

Proof: It will be helpful to write $C = C_{t_0, t_1} := (t_1 - t_0)B(A + (t_0 + t_1)B)^{-1}$ so that $X(\lambda) := \frac{\lambda}{1-\lambda}[I + \lambda C]^{-1}[I - (1 - \lambda)C]$. Suppose that $X(\lambda) = \frac{\lambda}{1-\lambda}kI$ where $k = k(t_0, t_1, \lambda)$ is some scalar function. Then

$$\begin{aligned} I - (1 - \lambda)C &= k(I + \lambda C) \\ (1 - k)I &= (k\lambda + 1 - \lambda)C \end{aligned}$$

which implies that C is some (possibly zero) multiple of I . By the definition of C this implies that B is a multiple of A . As observed before (page 31) this reduces to the straight line case.

On the other hand, if $BA^{-1}B = 0$, then $C = (t_1 - t_0)BA^{-1}$ and hence $X(\lambda) = \frac{\lambda}{1-\lambda}(I - (t_1 - t_0)BA^{-1})$, while $T := \frac{t_0}{t_1}(A + t_0B)(A + t_1B)^{-1} = \frac{t_0}{t_1}(I - (t_1 - t_0)BA^{-1})$. So $X(\lambda)$ and T are parallel. A simple 2×2 counterexample to the converse statement is given by

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & B &= \begin{pmatrix} 1/4 & 0 \\ 0 & -1/3 \end{pmatrix} \\ t_0 &= -8/9 & t_1 &= 2/9 & \lambda &= 1/2 \end{aligned}$$

which gives

$$X(\lambda) = \begin{pmatrix} 5/7 & 0 \\ 0 & 19/14 \end{pmatrix} \quad T = -4 \begin{pmatrix} 14/19 & 0 \\ 0 & 7/5 \end{pmatrix}. \quad \square$$

Corollary 39 (Nikodym result for $BA^{-1}B = 0$). *Nikodym sets of curves of the form (2.7) with $BA^{-1}B = 0$ have Minkowski dimension at least $\frac{n-1}{2-\varepsilon} + 1 \approx 0.5969n + 0.403$, where ε is the smallest root of $\varepsilon^3 - 6\varepsilon^2 + 8\varepsilon - 2$.*

Proof: In this case, for all t_0, t_1 the sumsets are just the scalar ones $\mathcal{A} + \frac{\lambda}{1-\lambda}\mathcal{B}$. Clearly by choosing suitable heights these can be any scalars we like, so this follows immediately from Katz and Tao's sumset result (Theorem 33). \square

Many other families of curves admit some good bound for the Nikodym sets, however. To use Katz and Tao's simple three-slice estimate with $\varepsilon = 1/6$

(Lemma 30), all we require is that there exist $t_0, t_1 \in [-1, 1] \setminus \{0\}$ and $\lambda \in (0, 1) \setminus \{\frac{t_0}{t_0 - t_1}\}$ such that $X(\lambda) = T$. These equations are difficult to solve, but where the matrix B is invertible, or where one of A, B is positive definite so that we may assume that they are diagonal and hence commuting, we can simplify the problem.

Corollary 40. *Suppose that either B is invertible or that one of A, B is positive definite, and that $A^{-1}B$ satisfies a quadratic polynomial. If the heights t_0, t_1 and $t_2 := (1 - \lambda)t_0 + \lambda t_1$ can be chosen so that certain functions of the heights equal the coefficients of the quadratic, then the corresponding curved Nikodym sets have Minkowski dimension at least $\frac{6n-6}{11}$. This choice of heights cannot be made unless the sum of the reciprocals of the eigenvalues is less than 3 in modulus.*

Proof: By multiplying both sides of the equation $X(\lambda) = T$ by appropriate matrices we find that in these cases, $X(\lambda) = T$ if and only if $A^{-1}B$ satisfies the following quadratic equation:

$$\begin{aligned} 0 &= [(t_0^2 + t_1^2)\lambda(\lambda(t_1 - t_0) + 2t_0) - t_0^2(t_0 + t_1)] (A^{-1}B)^2 \\ &\quad - [((1 - \lambda)t_0 + \lambda t_1)((1 - \lambda)t_0 + \lambda t_1 + t_0 + t_1)] (A^{-1}B) - ((1 - \lambda)t_0 + \lambda t_1) \\ &=: \left[\frac{t_0^2 + t_1^2}{t_1 - t_0} (t_2^2 - t_0^2) - t_0^2(t_0 + t_1) \right] (A^{-1}B)^2 - t_2(t_0 + t_1 + t_2)(A^{-1}B) - t_2 \end{aligned}$$

where we define $t_2 := (1 - \lambda)t_0 + \lambda t_1$. Thus for these cases, we require that $A^{-1}B$ should have at most two eigenvalues. Moreover, if the eigenvalues are distinct, then $A^{-1}B$ must be diagonalisable (over \mathbb{C}), while in the case of one repeated eigenvalue the Jordan normal form of $A^{-1}B$ must contain only 1×1 and 2×2 blocks. By considering the ratio of the last two coefficients we find that the sum of the reciprocals of the eigenvalues must be equal to $-(t_0 + t_1 + t_2) \in (-3, 3)$, which imposes further restriction on $A^{-1}B$.

Once we have found heights so that the quadratic is satisfied, the result follows from Lemmas 30 and 32. \square

Certainly some suitable A, B exist, for example

$$\begin{array}{lll} A = \begin{pmatrix} 4 & 1 \\ 1 & -13/2 \end{pmatrix} & B = \begin{pmatrix} -2 & 4 \\ 4 & 1 \end{pmatrix} & A^{-1}B = \begin{pmatrix} -1/3 & 1 \\ -2/3 & 0 \end{pmatrix} \\ t_0 = -1 & t_1 = 1 & \lambda = 3/4 \end{array}$$

for which the polynomial $-\frac{3}{4}(A^{-1}B)^2 - \frac{1}{4}(A^{-1}B) - \frac{1}{2}I = 0$ is indeed satisfied. Also $t_0 + t_1 + t_2 = -1 + 1 + 1/2 = 1/2$, while the sum of the reciprocals of the eigenvalues is $6/(-1 + \sqrt{23}i) + 6/(-1 - \sqrt{23}i) = -1/2$ as expected. However, it seems very difficult to describe concisely the set of all pairs A, B which work.

6.4.2 Non-scalar matrices

We now apply the new matrix sumset results of Section 6.3.2 to theakeya and Nikodym problems for curves.

First of all we note that the parabolas that have been most successful thus far in the Nikodym case are not at all amenable to this method of attack in theakeya case.

Theorem 41. *For aakeya set of curves of the form (2.7) with $BA^{-1}B = 0$ we cannot prove any non-trivial bound by sumset methods.*

Proof: If $BA^{-1}B = 0$, then we can calculate

$$X(\lambda) = \frac{\lambda}{1-\lambda}(I + (t_0 - t_1)BA^{-1}).$$

So we would need a sumset result where the X_j were all multiples of each other but not of I . But we have already seen in Theorem 36 that in such a case no non-trivial estimate can hold. \square

This is unsurprising, since Bourgain’s “worst case” example (Theorem 5) belongs to this class. However it is interesting that curves that are well behaved for Nikodym should not do so forakeya. We shall discuss this in the final chapter.

We now address the problem of realising the matrices that gave a positive answer to Question 4 as slices through sets. First let us consider the four-slice result (Theorem 37). For theakeya case, we need conditions on the curves (in terms of A and B) that guarantee the existence of $t_0, t_1 \in [-1, 1]$ and $\lambda, \mu \in (0, 1)$ such that $X(\lambda) - X(\mu) = I$.

Unfortunately, in many cases this cannot be done. This is hardly surprising, since for fixed A, B , we are trying to satisfy $(n-1)^2$ equations with only four unknowns. However, allowing A, B to vary gives a great many more unknowns, even when we assume that $A = I$ and B is in Jordan normal form. But the non-linearities involved cause difficulty as we shall see.

Theorem 42. *If the matrix C defined on page 86 is nilpotent, then $X(\lambda) - X(\mu)$ is never equal to the identity.*

Proof: Let k be the highest power of C that is non-zero. Then $[I + \lambda C]^{-1} = \sum_0^k (-1)^n \lambda^n C^n$, and hence

$$X(\lambda) - X(\mu) = \left(\frac{\lambda}{1-\lambda} - \frac{\mu}{1-\mu} \right) I + \sum_1^k (-1)^n C^n \left(\frac{\lambda^n}{1-\lambda} - \frac{\mu^n}{1-\mu} \right).$$

If this equals the identity, then some linear combination of I, C, C^2, \dots, C^k is zero. But this cannot happen because the minimum polynomial of a nilpotent matrix is x^{k+1} . \square

This cuts down the list of possible candidates for A, B . Note in particular that all matrices with $BA^{-1}B = 0$ make C nilpotent for every choice of t_0, t_1 . Obviously invertible matrices B are not ruled out, and nor are diagonal matrices, and whenever either A or B is positive definite we can fix A, B and hence C to be diagonal by a change of coordinates.

Proposition 43. *If B is invertible, or C is diagonal, then a necessary and sufficient condition for $X(\lambda) - X(\mu) = I$ is that C satisfies the following quadratic:*

$$\lambda\mu C^2 + \left(2\lambda + \lambda\mu\left(\frac{1}{1-\mu} - \frac{1}{1-\lambda}\right)\right)C - \left(\frac{\lambda}{1-\lambda} - \frac{\mu}{1-\mu} - 1\right)I = 0 \quad (6.6)$$

Proof: By Hörmander's criterion, B is invertible if and only if C is. Now $X(\lambda) = \frac{\lambda}{1-\lambda}(I + \lambda C)^{-1}(I + \lambda C - C)$ so we get

$$\begin{aligned} I &= X(\lambda) - X(\mu) \\ &= \left(\frac{\lambda}{1-\lambda} - \frac{\mu}{1-\mu}\right)I - \left(\frac{\lambda}{1-\lambda}(I + \lambda C)^{-1} - \frac{\mu}{1-\mu}(I + \mu C)^{-1}\right)C. \end{aligned}$$

Then if C is invertible we can multiply on the left by $C(I + \lambda C)$ and on the right by $C^{-1}(I + \mu C)$. If C is diagonal then everything commutes so we just multiply by $(I + \lambda C)(I + \mu C)$. After rearranging the result follows. \square

This is significant because a “generic” $(n-1) \times (n-1)$ matrix does not satisfy a polynomial of degree less than $n-1$: To satisfy a quadratic it must either be diagonalisable with at most two distinct eigenvalues, or have one repeated eigenvalue and have Jordan normal form consisting only of 1×1 and 2×2 blocks. Before we go on to look at those eigenvalues, let us find out more about the solutions of (6.6).

Suppose that C has two eigenvalues h and k . These are either real or form a complex conjugate pair, and so both their sum and their product are real.

Theorem 44. (i) *If $h + k < -2(1 + \sqrt{2})$, or if $h, k \in \mathbb{R}$ and at least one is less than -1 , then h, k are the roots of equation (6.6) for some $\lambda, \mu \in (0, 1)$.*

(ii) *If both h and k are real and have modulus at most 1, then there are no $\lambda, \mu \in (0, 1)$ such that h, k are the roots of equation (6.6).*

Proof: We obtain two simultaneous equations by considering the sum and product of the roots of (6.6).

$$\frac{1}{\lambda\mu} - \frac{1}{\mu(1-\lambda)} + \frac{1}{\lambda(1-\mu)} = hk \quad (6.7)$$

$$\frac{2}{\mu} + \frac{1}{1-\mu} - \frac{1}{1-\lambda} = -(h+k) \quad (6.8)$$

Now (6.8) is linear in λ so we solve it to obtain

$$\lambda = 1 - \frac{\mu}{\frac{2-\mu}{1-\mu} + \mu(h+k)}.$$

Of course this needs to lie in $(0, 1)$. Tedious calculation shows that in the case $h+k > -2(1+\sqrt{2})$ it does so for all $\mu \in (0, 1)$. For $h+k < -2(1+\sqrt{2})$ it does so provided we take

$$\mu \in \left(0, \frac{2 - (h+k) - \sqrt{(h+k+2)^2 - 8}}{2(1-h-k)}\right). \quad (6.9)$$

Next we substitute this expression for λ back into (6.7). After rearranging we obtain an equation which is quartic in μ and quadratic in h and k . With the help of MAPLE we express it as

$$0 = [hk(h+k-1)]\mu^4 + [2hk + (h+k)((h+k)^2 - hk - 1)]\mu^3 + [(h+k)(4 - (h+k)) - 1 - 2hk]\mu^2 + 4(1-h-k)\mu - 4 \quad (6.10)$$

$$= -[\mu^2(1-\mu)(\mu k + 1)]h^2 + [\mu(\mu^2 k - \mu k + \mu - 2)(\mu k - \mu + 2)]h + (\mu^2 k - \mu k + \mu - 2)(\mu k - \mu + 2) \quad (6.11)$$

$$=: q(\mu, h, k).$$

Note that this is a real-valued function of μ . For the first part we use a naïve approach via the intermediate value theorem. Setting $\mu = 0$ gives -4 , while $\mu = 1$ gives $-(h+1)(k+1)$. If this is positive, which can happen only when h and k are real with exactly one being less than -1 , then the intermediate value theorem guarantees a solution for μ . Setting $\mu = \frac{-2}{k-1}$, which we may provided that $k \in \mathbb{R}$ and $k < -1$, leaves only the h^2 term, so that $q = -\mu^2(1-\mu)(\mu k + 1)h^2$ which is easily seen to be positive. By symmetry the same happens if $h < -1$. If we instead substitute in the right hand endpoint from equation (6.9) (which is allowable if and only if $h+k < -2(1+\sqrt{2})$) we obtain

$$\frac{(h+k)^2 + (h+k-2)\sqrt{(h+k+2)^2 - 8}}{2(h+k-1)^2}$$

which by more tedious rearranging is seen to be positive for all h, k .

For the second assertion we show that the maximum of the function q over the region $(\mu, h, k) \in [0, 1] \times [-1, 1] \times [-1, 1]$ is zero, and moreover that this is attained only for $\mu = 1$. For interior maxima we use the version (6.11) of the equation as a quadratic in h . Its stationary point (a maximum) occurs at

$$h = \frac{(\mu^2 k - \mu k + \mu - 2)(\mu k - \mu + 2)}{2\mu(1-\mu)(\mu k + 1)}.$$

Now if $|k| < 1$ then the denominator is positive, so that the whole fraction will be less than -1 if

$$(\mu^2 k - \mu k + \mu - 2)(\mu k - \mu + 2) < -2\mu(1 - \mu)(\mu k + 1)$$

which rearranges to

$$\mu^2(1 - \mu)k^2(3(1 - \mu)^2 + 1)(\mu k + 1) > 0.$$

So there is no zero of $\frac{\partial q}{\partial h}$ in the region, except perhaps when $\mu = 1$ and $k = -1$. We find that $q(1, h, -1) \equiv 0$. Now to check the other boundaries:

$$\begin{aligned} q(\mu, 1, k) &= -k^2\mu^2(1 - \mu^2) - 2(\mu k + 1)(2 - \mu^2) \\ q(\mu, -1, k) &= -k^2\mu^2(1 - \mu)^2 - 2(\mu k + 1)(1 - \mu)((1 - \mu)^2 + 1) \end{aligned}$$

Both of these are clearly non-positive, and give zero only at $(1, h, -1)$ as we have already seen, and at $(1, 1, k)$. \square

We must now investigate the eigenvalues of C . At this point it is helpful to change coordinates so that $A = I$ and B is no longer assumed to be symmetric (see page 32). Since $C := (t_0 - t_1)B(I + (t_0 + t_1)B)^{-1}$ it is easy to check that e_B is an eigenvalue of B if and only if e_C is an eigenvalue of C , where

$$e_C = \frac{e_B(t_1 - t_0)}{1 + (t_0 + t_1)e_B}.$$

The non-degeneracy criterion tells us about the real eigenvalues of C .

Lemma 45. *All the real eigenvalues of C must lie in $[-1, 1]$.*

Proof: Suppose not. Then there exists $e_C \in \mathbb{R} \setminus [-1, 1]$ and $v \in \mathbb{R}^{n-1}$ with $Cv = e_C v$. Write $u = (A + (t_0 + t_1)B)^{-1}v$. This gives

$$\begin{aligned} (t - t_0)Bu &= e_C(A + (t_0 + t_1)B)u \\ \left[A + (t_0 + t_1)B - \frac{1}{e_C}(t_1 - t_0)B \right] u &= 0 \\ \det \left[A + \left((1 - \frac{1}{e_C})t_0 + (1 + \frac{1}{e_C})t_1 \right) B \right] &= 0. \end{aligned}$$

But $((1 - \frac{1}{e_C})t_0 + (1 + \frac{1}{e_C})t_1) \in [-2, 2]$ so this contradicts (2.2'). \square

Combining this with our observations above tells us that we cannot achieve $X(\lambda) - X(\mu) = I$ if one of the original A, B is positive definite, or if both are invertible and $A^{-1}B$ has real eigenvalues.

However, if both are invertible and the eigenvalues are complex conjugate, then it sometimes can be achieved.

Theorem 46. *Suppose that B is invertible and that $A^{-1}B$ is diagonalisable over \mathbb{C} and has only two eigenvalues $\alpha \pm \beta i$. Then if either α is sufficiently large and negative ($\alpha < -\frac{1+\sqrt{2}}{2}$ will do) or β is large compared to α , there is a lower bound of $\frac{4n+3}{7}$ for the curved Kakeya sets associated to A, B .*

Proof: We have seen that this holds if we can make $h + k < -2(1 + \sqrt{2})$. But $h + k$ is simply twice the real part of the eigenvalues of C , so we require

$$(t_1 - t_0) \frac{\alpha + (t_0 + t_1)(\alpha^2 + \beta^2)}{1 + 2(t_0 + t_1)\alpha + (t_0 + t_1)^2(\alpha^2 + \beta^2)} < -(1 + \sqrt{2}).$$

It helps to write $t_1 = 1 - 2\varepsilon, t_0 = -1 + \varepsilon$, where $\varepsilon < 2/3$ may be taken as small as we wish. The inequality becomes

$$\frac{\alpha(1 - \varepsilon\alpha) - \varepsilon\beta^2}{(1 - \varepsilon\alpha)^2 + \varepsilon^2\beta^2} < -\frac{1 + \sqrt{2}}{2 - 3\varepsilon}$$

Clearly this is satisfied for small ε and large β : Choosing $\varepsilon \lesssim 1/|\alpha|$ shows that $\beta \gtrsim |\alpha| + 1$ will work. Alternatively if $\alpha < -\frac{1+\sqrt{2}}{2}$ then we simply need to take ε very small. In both of these cases, we have in fact found a whole family of solutions for varying ε so there is no problem with using the argument about varying the heights of the planes which gave the extra $+1$ for the dimension bound in Lemma 32. \square

So we get a non-trivial result in some cases, although it is not easy to give the criteria any geometric interpretation.

However, about the case where B is not invertible, or where $A^{-1}B$ has more than two eigenvalues or two real ones, we cannot say anything other than that the above proof will not work.

We have not yet considered using four slices in the Nikodym case. This is more complicated, because we require $X(\lambda) - X(\mu) = T$ instead of I , which means that we cannot write this in terms of C and so we must look at all four variables t_0, t_1, λ, μ together, rather than in two stages as we did above. By the methods already used, we can show that if A and B are diagonal or invertible then $A^{-1}B$ must satisfy a cubic equation, and that the reciprocals of the roots (the eigenvalues of $A^{-1}B$) must have the same sum as minus the heights of the slices, as we found in Section 6.4.1. As one would expect, it is difficult to say anything more than that explicitly.

So what hope is there for the use of arithmetic methods? If we still want to use only four slices for cases not covered above, then we shall have to prove a new sumset result, that is, find a more flexible condition than $X(\lambda) - X(\mu) = I$ which guarantees that the difference set is not too much larger than the two original

sets and their $X(\lambda)$ and $X(\mu)$ matrix sumsets. Or we could instead look at using more slices—the techniques in [23] suggest that relations like

$$0 = X(\lambda) - X(\mu) + X(\nu)^{-1}X(\lambda)$$

$$\text{or } 0 = X(\lambda) - X(\mu) + X(\nu)^{-1}X(\lambda) - X(\kappa)^{-1}X(\mu)$$

would suffice. But of course these lead to higher degree polynomials in more variables which make it harder to compute sufficient conditions for suitable solutions to exist.

Chapter 7

Discussion

Instead of taking drugs I just start a new maths problem.
Tony Carbery

7.1 Summary

We now pause briefly to summarise the positive results we have proved. First of all we defined curved Kakeya and Nikodym sets and related maximal functions, and proved that estimates for Hörmander-type oscillatory integrals imply L^p bounds for the maximal functions, and lower bounds for the dimension of the sets. We then focused our attention on a special class of curves, namely the quadratic ones given by (2.7). Without further assumptions we were able to show that curved Kakeya and Nikodym sets can have measure zero, and that the so-called “trivial bound” (that is, the optimal bound from $L^{\frac{n+1}{2}}$) must hold for both the Kakeya and Nikodym maximal functions.

Turning to non-trivial results, we obtained an $L^n \rightarrow L^n$ bound of order $\delta^{-\frac{n-2}{2n}}$ for the curved Nikodym maximal function, implying the lower bound of $\frac{n+2}{2}$ for the Hausdorff dimension of the sets, in the special case where the coefficients A and B in the curves satisfy $BA^{-1}B = 0$. This result was proved using geometric arguments due to Katz.

Finally we considered arithmetic methods. For Nikodym sets, we were able to equal the best known bound for the straight line case (approximately $0.5969n + 0.403$) again in the case $BA^{-1}B = 0$, and obtain $\frac{6n-6}{11}$ in a few other cases that we did not attempt to describe explicitly, noting only the necessary condition that $A^{-1}B$ should satisfy a quadratic polynomial. In the Kakeya case, we were forced to use *matrix* sumset methods, and thus obtained the lower bound of $\frac{4n+3}{7}$ for sets where $A^{-1}B$ satisfies a particular quadratic. These we did attempt to describe explicitly, showing that $A^{-1}B$'s being a diagonalisable matrix with two complex conjugate eigenvalues satisfying certain size conditions is sufficient.

Conditions of the latter kind seem rather *ad hoc*, and disappointingly devoid of obvious geometric content. However, the condition $BA^{-1}B = 0$ is more interesting, as we now show.

7.2 The condition $BA^{-1}B = 0$

This condition, which seems to make the curved Nikodym problem particularly amenable to known methods of proof, has a surprising link with Bourgain's "worst case" example for the curved Kakeya problem.

That example had $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which clearly satisfy the condition. This is no accident: The following converse is also true:

Theorem 47. *If $BA^{-1}B = 0$, then the corresponding Kakeya sets can have dimension as low as $n - \text{rank}(A^{-1}B)$. In particular, in odd dimensions, the trivial lower bound of $\frac{n+1}{2}$ can be attained, while in even dimensions there can be sets with dimension at least as low as $\frac{n+2}{2}$.*

Proof: Let us without loss assume that $A = I$ and that B is in Jordan normal form. The criterion is then simply $B^2 = 0$, which holds if and only if B consists only of 1×1 and 2×2 Jordan blocks with eigenvalue 0. The rank of B is the number of 2×2 blocks, which can be at most $\frac{n-1}{2}$ if n is odd, and $\frac{n-2}{2}$ if n is even. Denote the rank of B by r .

We now proceed as on page 51. The curves $\Gamma_y(\omega)$ are given by

$$\begin{pmatrix} \omega_1 - 2ty_1 - 2t^2y_2 \\ \omega_2 - 2ty_2 \\ \vdots \\ \omega_{2i-1} - 2ty_{2i-1} - 2t^2y_{2i} \\ \omega_{2i} - 2ty_{2i} \\ \vdots \\ \omega_{2r-1} - 2ty_{2r-1} - 2t^2y_{2r} \\ \omega_{2r} - 2ty_{2r} \\ \omega_{2r+1} - 2ty_{2r+1} \\ \vdots \\ \omega_{n-1} - 2ty_{n-1} \\ t \end{pmatrix}$$

so that if we choose $\omega_{2i-1} = 0$ and $\omega_{2i} = -2y_{2i-1}$ for $i = 1, \dots, r$ and arbitrary thereafter, we find that $x_{2i-1} = x_{2i}x_n$ for $i = 1, \dots, r$. So the curves have been translated to lie in a surface of dimension $n - r$. \square

Clearly we could also modify the proof of Theorem 5 to show that the oscillatory integral operators with phase $\varphi(x, y) = x' \cdot y + x_n y^\top A y + x_n^2 y^\top B y$ where

which is exactly Restriction for the paraboloid but for the last term, which, as we know, causes the completely degenerate behaviour where no non-trivial result is possible for the set dimension, maximal function, or oscillatory integral operator.

These observations are rather striking, and it is not at all clear what to conclude. It does suggest however that although up until now Restriction/Keakeya and Bochner-Riesz/Nikodym have been thought of as essentially the same, they might be better described as dual in some way, or even opposite. This idea is not so strange when one remembers that curvature of the surface in question is *good* when considering Restriction (since it causes decay of the Fourier transform) but *bad* for Bochner-Riesz (Bochner-Riesz for squares is trivial).

This also shows the importance of Carbery's transformation

$$(x', x_n) \mapsto (x'/x_n, 1/x_n)$$

which relates the two classes of problems. One might speculate that whenever a family of curves has a negative answer for (say) the Keakeya problem, then both the Nikodym problem for the same curves, and equivalently the Keakeya problem for the transformed curves, has a positive answer. So in Figure 2.2 the problems in the second box are opposite to those in the third. This would leave straight lines as an overlapping middle case, the only family that this transformation leaves unchanged.

7.3 The future

Clearly this thesis has uncovered only a small patch of a large uncharted area, and much is left to be done. First of all, to what extent do our results depend on the special quadratic type of phase we decided to consider? An extension of the work to phases with higher terms in x_n is probably routine, but extension to higher terms in x' and y may well not be. The difficulty is that in those cases, we can no longer parametrise our curves by the height; nor do we have the linearity in y and ω which was used to simplify much of our reasoning.

There are still geometric methods used in the straight line case that we have not yet tried. Although we saw in Chapter 5 some reasons for pessimism about the sticky/plany/grainy ideas of Katz, Laba and Tao [20, 24], there is also the paper of Schlag [31] to try. He considers lines meeting three given ones, instead of two as in Katz's method. And of course, being a new problem, the matrix sumset idea offers an unknown number of possibilities for attacking curved Keakeya sets.

Also, thinking of the straight line problem, our results suggest that the idea of intersecting properly rather than tangentially is key. Hörmander's criterion

(2.2) merely demands that curves intersect properly, and we have seen that this is not enough. But in the straight line case, not only the lines themselves, but all higher-dimensional manifolds formed from them intersect properly—they are all hyperplanes, never curved surfaces. We suggest that this fact needs to be used somehow, and for Hörmander’s general question perhaps higher-order non-degeneracy conditions on the phase are required.

However, it is probably the hint at a new understanding of the relationship between Restriction and Bochner-Riesz that is most important, and holds the most promise for future work.

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